# CANONICAL MODELS FOR ARITHMETIC $(1;e)$-CURVES

JEROEN SIJSLING

ABSTRACT. The Fuchsian groups of signature $(1;e)$ are the simplest class of Fuchsian groups for which the calculation of the corresponding quotient of the upper half plane presents a challenge. This paper considers the finite list of arithmetic $(1;e)$-groups. We define canonical models for the associated quotients by relating these to genus 1 Shimura curves. These models are then calculated by applying results on the $\mathfrak{p}$-adic uniformization of Shimura curves and Hilbert modular forms.

Let $\mathrm{GL}_2(\mathbb{R})^+$ be the group of invertible real two-by-two matrices with positive determinant, and let $\mathrm{PGL}_2(\mathbb{R})^+$ be the adjoint group of $\mathrm{GL}_2(\mathbb{R})^+$. Through Möbius tranformations, $\mathrm{PGL}_2(\mathbb{R})^+$ acts faithfully on the complex upper half plane $\mathcal{H}^+ = \mathcal{H}$ and the complex lower half plane $\mathcal{H}^- = \overline{\mathcal{H}}$. Consider a Fuchsian group $\Gamma \subset \mathrm{PGL}_2(\mathbb{R})^+$ that is arithmetic (as defined in [Tak75]). Then one can construct the Riemann surfaces $X^{\pm}(\Gamma) = \Gamma\backslash\mathcal{H}^{\pm}$. By definition, these curves allow a finite correspondence with some Shimura curve associated to an order in a quaternion algebra over a totally real field.

Canonical models for Shimura curves over number fields were first constructed in [Shi70]. The explicit determination of these models has received a fair amount of attention in recent years, for example in [Elk98], [GR06], [Mol10] and [Voi09a]. However, less effort seems to have been put into finding equations for more general arithmetic curves. By contrast, this paper will focus on a specific class of arithmetic curves, namely those coming from arithmetic Fuchsian groups $\Gamma$ whose signature equals $(1;e)$ for some natural number $e \geq 2$. Geometrically, for $\Gamma$ to have signature $(1;e)$ means that the quotients $X^{\pm}(\Gamma)$ are of genus 1 and that the branch loci of the projection maps $\mathcal{H}^{\pm} \to X^{\pm}(\Gamma)$ consist of a single point, above which ramification of index $e$ occurs.

The full classification of arithmetic $(1;e)$-groups is due to Takeuchi: by [Tak83, Theorem 4.1], there are 71 arithmetic $(1;e)$-groups up to $\mathrm{PGL}_2(\mathbb{R})$-conjugacy. Our motivation for studying arithmetic $(1;e)$-groups is that they are the next simplest type of arithmetic Fuchsian groups after triangle groups, being the only other type of cocompact Fuchsian group generated by two elements.

In Definition 2.3.3, we define canonical models for arithmetic $(1;e)$-curves by realizing these curves as Atkin–Lehner quotients of Shimura curves. In Section 4, we determine these canonical models in 56 cases; in the remaining cases, only the corresponding isogeny class is found.

Our main strategy, adapted from [DD08] and described in Section 3, is to first find a point on a classical modular curve $Y_0(p)$ that corresponds to the curve $X^{\pm}(\Gamma)$ by using [BZ]. Taking an appropriate twist in the associated geometric isomorphism class of elliptic curves, we obtain a conjectural model of $X^{\pm}(\Gamma)$. We attempt to prove the correctness of this model (or, if this fails, of its isogeny class) by invoking modular methods. In the fortunate 27 cases where $\Gamma$ is commensurable with a triangle group, it is also possible to determine canonical models for the curves $X^{\pm}(\Gamma)$ by taking appropriate twists of the models over $\mathbb{C}$ constructed in [Sij11].

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The organization of this paper is as follows. After fixing our notation, Sections 1 and 2 summarize the notions from the theory of quaternion algebras and Shimura curves that we shall need; moreover, we define canonical models for arithmetic $(1;e)$-curves at the end of Section 2. Section 3 is devoted to making explicit the results in [BZ] and describing how to search for a point corresponding to $X^{\pm}(\Gamma)$ on a classical modular curve $Y_0(p)$. In Section 4, we determine conjectural equations for the curves $X^{\pm}(\Gamma)$ associated to the $(1;e)$-curves in [Tak83, Theorem 4.1]; the correctness of these equations is proved in Section 5. The Appendix summarizes our results in the form of three tables.

We used the computer algebra system `Magma` ([BCP97]) to perform our calculations. The programs used for this paper can be found at [Sij10b].

Along with [Sij11], this paper summarizes the results in the author's Ph.D. thesis [Sij10a].

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## NOTATION

We denote the unit group of a ring $R$ by $R^{\times}$ and the cardinality of a set $S$ by $|S|$. Unless explicitly mentioned otherwise, our further notation is the following.

### 0.1. **Fuchsian groups.**

- $\Gamma$: an arithmetic Fuchsian group of signature $(1;e)$.
- $\widetilde{\Gamma}$: the lift of $\Gamma$ to $\mathrm{SL}_2(\mathbb{R})$ defined in [Tak83, Section 3].
- $\Gamma^{(2)} = \langle \gamma^2 : \gamma \in \Gamma \rangle$.
- $\widetilde{\Gamma}^{(2)} = \langle \widetilde{\gamma}^2 : \widetilde{\gamma} \in \widetilde{\Gamma} \rangle$.
- $\alpha, \beta$: the generators of $\Gamma$ given in [Tak83, Section 3].

### 0.2. **Number fields.**

- $F$: a totally real number field. When considering an arithmetic $(1;e)$-group $\Gamma$, this denotes the trace field $\mathbb{Q}(\mathrm{tr}(\widetilde{\Gamma}^{(2)}))$ of $\Gamma$ (as in [Tak83]).
- $\mathbb{Z}_F$: the ring of integers of $F$.
- $F^+$ (resp. $\mathbb{Z}_F^+$): the totally positive units of $F$ (resp. $\mathbb{Z}_F$).
- $F_{\mathfrak{p}}$ (resp. $\mathbb{Z}_{F,\mathfrak{p}}$): the completion of $F$ (resp. $\mathbb{Z}_F$) at a prime ideal $\mathfrak{p}$ of $\mathbb{Z}_F$.
- $\mathbb{Z}_{F,\mathfrak{p}}$: the ring of integers of $F_{\mathfrak{p}}$.
- $U_{\mathfrak{p}}^{(k_{\mathfrak{p}})}$: the open subset $\mathbb{Z}_{F,\mathfrak{p}}^{\times} \cap (1 + \mathfrak{p}^{k_{\mathfrak{p}}})$ of $F_{\mathfrak{p}}^{\times}$. We abbreviate $U_{\mathfrak{p}}^2 = (U_{\mathfrak{p}}^{(0)})^2$.
- $\widehat{F}$: the finite adèle ring over $F$.
- $\widehat{\mathbb{Z}}_F$: the integral closure of $\mathbb{Z}$ in $\widehat{F}$.
- $\mathrm{Cl}(N)$ (resp. $\mathrm{Cl}(N\infty)$): the ray class group $F^{\times} \backslash \widehat{F}^{\times} / N$ (resp. narrow ray class group $F^{+} \backslash \widehat{F}^{\times} / N$) associated to an open subgroup $N$ of $\widehat{F}^{\times}$.
- $\mathrm{Cl}(\prod \mathfrak{p}^{k_{\mathfrak{p}}})$ (resp. $\mathrm{Cl}(\prod \mathfrak{p}^{k_{\mathfrak{p}}} \infty)$): the ordinary (resp. narrow) ray class group associated to the open subgroup $\prod U_{\mathfrak{p}}^{(k_{\mathfrak{p}})}$ of $\widehat{F}^{\times}$. For example, $\mathrm{Cl}(\infty)$ is the narrow Hilbert class group of $F$.
- $F_{\prod \mathfrak{p}^{k_{\mathfrak{p}}}}$ (resp. $F_{\prod \mathfrak{p}^{k_{\mathfrak{p}}} \infty}$): the ordinary (resp. narrow) ray class field associated to $\prod U_{\mathfrak{p}}^{(k_{\mathfrak{p}})}$.
- $G_F$: the absolute Galois group $\mathrm{Gal}(\overline{F}|F)$ of $F$.

0.3. **Quaternion algebras.**

- $B$: a quaternion algebra over a totally real field $F$ satisfying

$$B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}) \times \mathbb{H} \times \cdots \times \mathbb{H}. \tag{0.1}$$

  When considering an arithmetic $(1;e)$-group $\Gamma$, this will denote the quaternion algebra $F[\widetilde{\Gamma}^{(2)}]$ over the trace field $F$ of $\Gamma$.
- $\iota$: the embedding $B \hookrightarrow M_2(\mathbb{R})$ corresponding to the decomposition (0.1). Abusing notation, this also denotes the infinite place $\iota : F \hookrightarrow \mathbb{R}$ of $F$ for which $B \otimes_{F,\iota} \mathbb{R} \cong M_2(\mathbb{R})$.
- $H$: a quaternion algebra over a totally real field satisfying

$$H \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H} \times \cdots \times \mathbb{H}. \tag{0.2}$$

- nrd: the reduced norm map of $B$.
- $\mathfrak{D}(B)$: the discriminant of $B$, *i.e.*, the product of the places of $F$ at which $B$ ramifies.
- $\mathfrak{D}(B)^f$: the finite part of $\mathfrak{D}(B)$.
- $I$: a lattice of $B$, *i.e.*, a projective $\mathbb{Z}_F$-submodule of $B$ of rank 4.
- $\mathcal{O}$: an order of $B$, *i.e.*, a lattice that is also a subring.
- $\mathbb{Z}_F[\Gamma^{(2)}]$: the quaternion order $\mathbb{Z}_F[\widetilde{\Gamma}^{(2)}]$ associated to a $(1;e)$-group $\Gamma$.
- $\widehat{B}$ (resp. $\widehat{I}$): the tensor product $B \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}}$ (resp. $I \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$).
- $B_{\mathfrak{p}}$ (resp. $I_{\mathfrak{p}}$): the tensor product $B \otimes_F F_{\mathfrak{p}}$ (resp. $I \otimes_{\mathbb{Z}_F} \mathbb{Z}_{F,\mathfrak{p}}$).
- $B^{\mathfrak{p}}$ (resp. $I^{\mathfrak{p}}$): the ring $\widehat{B} \cap \prod_{\mathfrak{q} \neq \mathfrak{p}} B_{\mathfrak{q}}$. (resp. $\widehat{I} \cap \prod_{\mathfrak{q} \neq \mathfrak{p}} I_{\mathfrak{q}}$).
- $B^+$ (resp. $\mathcal{O}^+$): the group of units of $B$ (resp. $\mathcal{O}$) whose reduced norm is totally positive.
- $B^1$ (resp. $\mathcal{O}^1$): the group of units of $B$ (resp. $\mathcal{O}$) whose reduced norm equals 1.
- $K$: a compact open subgroup of $\widehat{B}^\times$.
- $K_{\mathfrak{p}} = K \cap B_{\mathfrak{p}}$.
- $K^{\mathfrak{p}} = K \cap \prod_{\mathfrak{q} \neq \mathfrak{p}} B_{\mathfrak{q}}$.
- $\mathrm{Cl}(K)$: the narrow ray class group $\mathrm{Cl}(\mathrm{nrd}(K)\infty)$.
- $F_K$: the narrow ray class field associated to $\mathrm{nrd}(K)$.
- $\mathrm{Pic}_r(K)$ (resp. $\mathrm{Pic}_r(K\infty)$): the ordinary (resp. narrow) right Picard set of $K$. We abbreviate $\mathrm{Pic}_r(\mathcal{O}) = \mathrm{Pic}_r(\widehat{\mathcal{O}}^\times)$.
- $T(K\infty)$: the right narrow type set of $K$.
- $N(K)$: the normalizer of $K$ in $B^\times$.
- $a([\mathfrak{a}])$: the Atkin–Lehner involution associated to an ideal class $\mathfrak{a}$.

0.4. **Geometry.**

- $\mathcal{H}^+ = \mathcal{H}$: the complex upper half plane.
- $\mathcal{H}^- = \overline{\mathcal{H}}$: the complex lower half plane.
- $X^+(G)$ (resp. $X^-(G)$): the Riemann surfaces $G \backslash \mathcal{H}^+$ (resp. $G \backslash \mathcal{H}^-$) associated to a Fuchsian subgroup $G$ of $\mathrm{GL}_2(\mathbb{R})^+$ or $\mathrm{PGL}_2(\mathbb{R})$. We abbreviate $X(G) = X^+(G)$.
- $X^\pm(\mathcal{O}^+)$ (resp. $X^\pm(\mathcal{O}^1)$): the complex algebraic curves $\mathcal{O}^+ \backslash \mathcal{H}^\pm$ (resp. $\mathcal{O}^1 \backslash \mathcal{H}^\pm$). Here we view the adjoint groups $\mathcal{O}^+$ and $\mathcal{O}^1$ as subgroups of $\mathrm{GL}_2(\mathbb{R})^+$ through $\iota$.
- $J^\pm(G)$: the Jacobians $\mathrm{Jac}(X^\pm(G))$.
- $Y(K)$: the Shimura curve associated to $K$. We abbreviate $Y(\mathcal{O}) = Y(\widehat{\mathcal{O}}^\times)$.
- $Y_0^+(K)$ (resp. $Y_0^-(K)$): the component of $Y(K)$ containing $[+i, 1]$ (resp. $[-i, 1]$).
- $\mathrm{Sh}(K)$ (resp. $\mathrm{Sh}_0^\pm(K)$): the canonical model of $Y(K)$ (resp. $Y_0^\pm(K)$).
- $J(K)$ (resp. $J_0^\pm(K)$): the Jacobian $\mathrm{Jac}(\mathrm{Sh}(K))$ (resp. $\mathrm{Jac}(\mathrm{Sh}_0^\pm(K))$).

0.5. **Uniformization.**

- $\Omega_{\mathfrak{p}}$: the $\mathfrak{p}$-adic upper half plane.
- $T_{\mathfrak{p}}$: the $\mathfrak{p}$-adic Bruhat-Tits tree.
- $\mathrm{Sh}(K, \mathfrak{p})$: the integral model for $\mathrm{Sh}(K)$ over $\mathbb{Z}_{F,\mathfrak{p}}$ constructed in [BZ].
- $G(K, \mathfrak{p})$: the graph associated to $\mathrm{Sh}(K, \mathfrak{p})$.
- $V(G)$: the vertex set of a graph $G$.
- $OE(G)$: the oriented edge set of a graph $G$.
- $E(G)$: the edge set of a graph $G$.

0.6. **Miscellaneous.**

- e$n_e$ d$n_d$ D$n_D$ r (*e.g.* e2d1D6i): the labels defined in the Appendix.
- $w_d$: the algebraic number given by

$$w_d = \begin{cases} \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4; \\ \sqrt{d} & \text{otherwise.} \end{cases} \tag{0.3}$$

- $i$: the square root of $-1$.

## 1. Quaternionic preliminaries

1.1. **Quaternion orders.** For the general theory of quaternion algebras and quaternion orders, we refer to [Vig80]. We need a few additional notions.

Let $\mathcal{O}$ be an order of $B$ and let $\mathcal{O}(1)$ be a maximal order containing $\mathcal{O}$. Then the quotient $\mathcal{O}(1)/\mathcal{O}$ is a finite $\mathbb{Z}_F$-module. Hence there exist $\mathbb{Z}_F$-ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ such that

$$\mathcal{O}(1)/\mathcal{O} \cong \prod_{k=1}^{n} \mathbb{Z}_F/\mathfrak{a}_k, \tag{1.1}$$

We define the *level* $\mathfrak{L}$ of $\mathcal{O}$ to be the $\mathbb{Z}_F$-ideal $\prod_k \mathfrak{a}_k$. It does not depend on the choice of $\mathcal{O}(1)$ as $\mathfrak{L}^2 = \mathrm{disc}(\mathcal{O}(1))/\mathrm{disc}(\mathcal{O})$, *cf.* [Vig80, Section III.5.A].

An *Eichler order* of $B$ is the intersection of two maximal orders of $B$.

**Proposition 1.1.1.** *Let $\mathcal{O}$ be an Eichler order of $B$. Then the following statements hold.*

(i) *At primes $\mathfrak{p}$ where $B$ ramifies, $\mathcal{O}_{\mathfrak{p}}$ equals the unique maximal order of $B_{\mathfrak{p}}$.*
(ii) *At primes $\mathfrak{p}$ where $B$ splits, choose an isomorphism $B_{\mathfrak{p}} \cong M_2(F_{\mathfrak{p}})$. Then there is a unique integer $n \in \mathbb{Z}_{\geq 0}$ such that under this isomorphism, $\mathcal{O}_{\mathfrak{p}}$ is conjugate to the suborder*

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_{F,\mathfrak{p}}) : c \in \mathfrak{p}^n \right\} \tag{1.2}$$

*of $M_2(F_{\mathfrak{p}})$. The exponent of $\mathfrak{p}$ in the level of $\mathcal{O}$ equals $n$.*

*Conversely, let $\mathcal{O}$ be an Eichler order maximal at a prime $\mathfrak{p}$ and let $\mathcal{O}(\mathfrak{p})$ be a level $\mathfrak{p}$ suborder of $\mathcal{O}$. Then $\mathcal{O}(\mathfrak{p})$ is an Eichler order.*

*Proof.* For the first part of the Proposition, see Lemme 1.5, Théorème II.2.3 and Lemme II.2.4 in [Vig80]. The converse statement is in [Eic55]. □

We now consider more general orders than Eichler orders. In our calculations in Section 4, we will encounter orders $\mathcal{O}$ with the property that there exist a maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$ and a squarefree ideal $\mathfrak{N}$ such that $\mathfrak{N}\mathcal{O}(1) \subset \mathcal{O}$. The following two propositions give a local description of such orders (*cf.* [HPS89]).

Let $F$ be a non-archimedean local field with valuation ring $\mathbb{Z}_F$ and let $\mathfrak{p}$ be the unique prime ideal of $\mathbb{Z}_F$, uniformized by $\pi$. Let $\kappa$ be the residue field $\mathbb{Z}_F/\mathfrak{p}$ and let $q = |\kappa|$. For $k \in \mathbb{N}$, we denote $U^{(k)} = \mathbb{Z}_F^\times \cap (1 + \mathfrak{p}^k)$, and we abbreviate $U^2 = (U^{(0)})^2$. Let $\varphi : \lambda \longrightarrow M_2(\kappa)$ be an embedding of the unique quadratic field extension $\lambda$ of $\kappa$ into the matrix ring $M_2(\kappa)$. Such an embedding exists (for example, view $\lambda$ as a vector space of dimension 2 over $\kappa$ and consider the left action of $\lambda$ on itself), and two such embeddings are conjugate by the Skolem-Noether Theorem ([Vig80, Théorème I.2.1]).

**Proposition 1.1.2.** *Let $B = M_2(F)$. Let $\mathcal{O}(1)$ be the maximal order $M_2(\mathbb{Z}_F)$ of $B$. Let $\mathcal{O} \subseteq \mathcal{O}(1)$ be an order such that $\mathfrak{p}\mathcal{O}(1) \subset \mathcal{O}$. Then up to conjugation by elements of $\mathcal{O}(1)^\times$, the order $\mathcal{O}$ is of exactly one of the following forms:*

(i) $\mathcal{O} = \mathcal{O}(1)$;

(ii) $\mathcal{O} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{O}(1) : c \equiv 0 \pmod{\mathfrak{p}} \right\}$;

(iii) $\mathcal{O} = \{x \in \mathcal{O}(1) : x \pmod{\mathfrak{p}} \in \varphi(\lambda)\}$;

(iv) $\mathcal{O} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{O}(1) : a \equiv d \pmod{\mathfrak{p}}, c \equiv 0 \pmod{\mathfrak{p}} \right\}$;

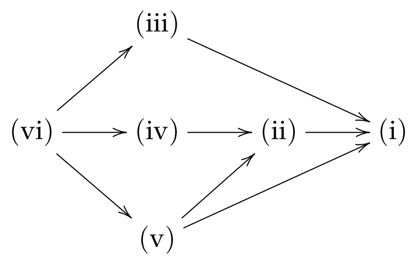
(v) $\mathcal{O} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{O}(1) : b \equiv c \equiv 0 \pmod{\mathfrak{p}} \right\}$;

(vi) $\mathcal{O} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{O}(1) : a \equiv d \pmod{\mathfrak{p}}, b \equiv c \equiv 0 \pmod{\mathfrak{p}} \right\}$;

*The following table summarizes the properties of the $\mathcal{O}$ above:*

| *Case* | *Level* | $[\mathcal{O}(1)^\times : \mathcal{O}^\times]$ | $\mathrm{nrd}(\mathcal{O}^\times)$ |
|---|---|---|---|
| *(i)* | $(1)$ | $1$ | $U^{(0)}$ |
| *(ii)* | $\mathfrak{p}$ | $q+1$ | $U^{(0)}$ |
| *(iii)* | $\mathfrak{p}^2$ | $(q-1)q$ | $U^{(0)}$ |
| *(iv)* | $\mathfrak{p}^2$ | $(q-1)(q+1)$ | $U^{(1)}U^2$ |
| *(v)* | $\mathfrak{p}^2$ | $q(q+1)$ | $U^{(0)}$ |
| *(vi)* | $\mathfrak{p}^3$ | $(q-1)q(q+1)$ | $U^{(1)}U^2$ |

*$\mathcal{O}$ is an Eichler order if and only if we are in case (i), (ii) or (v). Up to conjugation, the inclusion relations between these orders are as follows (except in the case $q = 2$, when the groups in (v) and (vi) are equal):*




*Proof.* This reduces to classifying the subalgebras of $\mathcal{O}(1)/\mathfrak{p}\mathcal{O}(1) \cong M_2(\kappa)$. To calculate the reduced norm groups, note that $\mathrm{nrd}(\mathcal{O}^\times) = U^{(1)}U^2$ in case (vi), and certainly $\mathrm{nrd}(\mathcal{O}^\times) = U^{(0)}$ in case (i) and (ii). As for the remaining cases, they follow because $\kappa[x]/(x^2)$ is the only algebra extension of $\kappa$ for which the norm to $\kappa$ does not surject in odd characteristic. □

*Remark* 1.1.3. Part (v) of the proposition shows that for an Eichler order $\mathcal{O}$, the quotient $\mathbb{Z}_F$-module $\mathcal{O}(1)/\mathcal{O}$ from (1.1) may depend on the choice of a maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$.

Let $A(\lambda)$ the $\kappa$-algebra whose underlying $\kappa$-vector space is given by $\lambda 1 \oplus \lambda u$ and whose multiplication is given by $(a_1 + b_1 u)(a_2 + b_2 u) = a_1 a_2 + (a_1 b_2 + b_1 a_2^q)u$.

**Proposition 1.1.4.** *Let $B$ be the ramified quaternion algebra over $F$ and let $\mathcal{O}(1)$ be the maximal order of $B$. Then there is an isomorphism of algebras*

$$\mathcal{O}(1)/\mathfrak{p}\mathcal{O}(1) \cong A(\lambda). \tag{1.3}$$

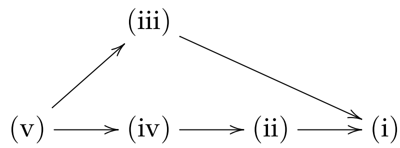
*Let $\mathcal{O} \subseteq \mathcal{O}(1)$ be an order such that $\mathfrak{p}\mathcal{O}(1) \subset \mathcal{O}$ and let $\mathcal{O}/\mathfrak{p}$ be the image of $\mathcal{O}$ under the isomorphism (1.3). Then up to conjugation by the elements of the groups $\mathcal{O}(1)^\times$ or $B^\times$, the subalgebra $\mathcal{O}/\mathfrak{p}$ of $A(\lambda)$ is of exactly one of the following forms:*

(i) *$\mathcal{O}/\mathfrak{p} = A(\lambda)$;*
(ii) *$\mathcal{O}/\mathfrak{p} = \kappa \oplus \lambda u$;*
(iii) *$\mathcal{O}/\mathfrak{p} = \kappa \oplus \kappa(a + bu)$, where $a$ is a fixed element of $\lambda - \kappa$ and $b$ runs through a set of representatives for $\lambda/\lambda^{\times(q-1)}$;*
(iv) *$\mathcal{O}/\mathfrak{p} = \kappa \oplus \kappa bu$, where $b$ runs through a set of representatives for $\lambda^\times/\lambda^{\times 2}$;*
(v) *$\mathcal{O}/\mathfrak{p} = \kappa \oplus \{0\}$.*

*The following table summarizes the properties of the $\mathcal{O}$ above:*

| *Case* | *Level* | $[\mathcal{O}(1)^\times : \mathcal{O}^\times]$ | $\mathrm{nrd}(\mathcal{O}^\times)$ |
|---|---|---|---|
| *(i)* | $(1)$ | $1$ | $U^{(0)}$ |
| *(ii)* | $\mathfrak{p}$ | $q+1$ | $U^{(1)}U^2$ |
| *(iii)* | $\mathfrak{p}^2$ | $q^2$ | $U^{(0)}$ |
| *(iv)* | $\mathfrak{p}^2$ | $q(q+1)$ | $U^{(1)}U^2$ |
| *(v)* | $\mathfrak{p}^3$ | $q^2(q+1)$ | $U^{(1)}U^2$ |

*The order of $B$ from case (ii) is the unique level $\mathfrak{p}$ order of $B$. Up to conjugation, the inclusion relations between these orders are as follows:*




*Proof.* The first part of the proposition is obtained by reducing the isomorphism in [Vig80, Corollaire I.1.7] modulo $\mathfrak{p}\mathcal{O}(1)$. Note that the algebra $\mathcal{O}(1)/\mathfrak{p}\mathcal{O}(1)$ is not central simple over $\kappa$. Classifying the requested suborders of $\mathcal{O}(1)$ once more reduces to calculating the subalgebras of $A(\lambda)$.

To show that $\kappa \oplus \lambda u$ is the unique level $\mathfrak{p}$ suborder of $A(\lambda, u)$, let $B$ be another $\kappa$-subspace of $A(\lambda, u)$ of codimension 1. Then the projection $\pi : V \to \lambda \oplus \{0\}$ is surjective. Let $x$ be an element of the non-trivial intersection $V \cap (\{0\} \oplus \lambda u)$. Then $vx = \pi(v)x$ for all $v \in V$. Since $\pi$ is surjective, $V$ contains the subspace $\{0\} \oplus \lambda u$. Now if $V$ were an order, it would also contain the subspace $\kappa \oplus \{0\}$. Hence because $V$ has codimension 1, we would then have $V = \kappa \oplus \lambda u$, contrary to our assumption.

The quadratic subalgebras of $\mathcal{O}/\mathfrak{p}$ are given by $Q = \kappa \oplus \kappa(a + bu)$ for some $a + bu \notin \kappa \oplus \{0\}$. The isomorphism class of $Q$ as a $\kappa$-algebra is determined by the reduced trace and norm of $a + bu$. Case (iii) is the case where $Q$ is a field, and in case (iv) $Q \cong k[x]/(x^2)$. The classification up to $\mathcal{O}(1)$-conjugacy follows from the fact that for $c \neq 0$ we have

$$\begin{aligned}(c + du)(a + bu)(c + du)^{-1} &= (c + du)(a + bu)(c^q - du)/c^{q+1} \\ &= a + bc^{-q+1}u.\end{aligned} \tag{1.4}$$

To conclude that the same holds up to $N(\mathcal{O}(1)^\times)$-conjugation, first note that if we let $\widetilde{u} \in \mathcal{O}(1)$ be a preimage of $u$, then $B^\times = \langle \mathcal{O}(1)^\times, \widetilde{u} \rangle$. If we let $\widetilde{a} + \widetilde{b}\widetilde{u}$ be an element of $\mathcal{O}$ lifting $a + bu$, then [Vig80, Corollaire II.1.7] shows that $\widetilde{u}(\widetilde{a} + \widetilde{b}\widetilde{u})\widetilde{u}^{-1}$ reduces to $a + b^q u$ in $A(\lambda, u)$. But $b^q = b \cdot b^{q-1}$, so we could have obtained the corresponding conjugate algebra equally well by conjugating with an element of $\mathcal{O}(1)^\times$. This concludes our illustration of case (iii).

Calculating the indices $[\mathcal{O}(1)^\times : \mathcal{O}^\times]$ is straightforward. As for the norm groups, note that if we let

$$\mathcal{O}(1)^{(1)} = \mathrm{Ker}(\mathcal{O}(1)^\times \longrightarrow (\mathcal{O}(1)/\mathfrak{p}\mathcal{O}(1))^\times), \tag{1.5}$$

then $\mathcal{O}(1)^{(1)} \subseteq \mathcal{O}^\times$ for all $\mathcal{O}$ as above. Let $L$ be the unramified quadratic extension of $F$. Then $U_L^{(1)} \subseteq \mathcal{O}(1)^{(1)}$ by [Vig80, Corollaire II.1.7]. Now $\mathrm{nrd}(U_L^{(1)}) = U_F^{(1)}$ by [Neu99, Corollary V.1.2], which quickly yields the final column of the table above. □

1.2. **Norms.** Let $F$ be a totally real number field and let $B$ be a quaternion algebra over $F$. Then we denote

$$F_B^\times = \left\{ x \in F^\times : \iota(x) > 0 \text{ for all } \iota \text{ dividing } \mathfrak{D}(B)^\infty \right\}, \tag{1.6}$$

and $\mathbb{Z}_{F,B}^\times = \mathbb{Z}_F^\times \cap F_B^\times$.

**Theorem 1.2.1.** *Let $B$ be an indefinite quaternion algebra over totally real number field $F$. Let $K$ be a compact open subgroup of $\widehat{B}^\times$ and let $\mathcal{O}$ be an Eichler order of $B$.*

(i) *We have* $\mathrm{nrd}(B^\times) = F_B^\times$.
(ii) *The group* $\mathrm{nrd}(K)$ *is an open subgroup of* $\widehat{\mathbb{Z}}_F^\times$. *Moreover,*

$$\mathrm{nrd}(K \cap B^\times) = \mathrm{nrd}(K) \cap \mathbb{Z}_{F,B}^\times. \tag{1.7}$$

(iii) *We have* $\mathrm{nrd}(\widehat{\mathcal{O}}^\times) = \widehat{\mathbb{Z}}_F^\times$ *and* $\mathrm{nrd}(\mathcal{O}^\times) = \mathbb{Z}_{F,B}^\times$.

*Proof.* Part (i) is the Eichler norm theorem, and part (ii) can be proved as in [Vig80, Proposition III.5.8]. Part (iii) then follows from Proposition 1.1.1(ii). □

*Remark* 1.2.2. For more general orders $\mathcal{O}$, the image $\mathrm{nrd}(\widehat{\mathcal{O}}^\times) \subseteq \widehat{\mathbb{Z}}_F^\times$ can be determined by a local calculation using Hensel's lemma at the primes $\mathfrak{p}$ where $\mathcal{O}$ is not maximal. Note that since $\mathcal{O}_\mathfrak{p}$ contains $\mathbb{Z}_{F,\mathfrak{p}}$, the norm group $\mathrm{nrd}(\mathcal{O}_\mathfrak{p}^\times)$ contains the open subgroup $\mathbb{Z}_{F,\mathfrak{p}}^{\times 2}$ of $\mathbb{Z}_{F,\mathfrak{p}}^\times$. We refer to [Sij10b] for some elaborate calculations.

1.3. **Indices.** This section will calculate indices associated to an inclusion $K' \subseteq K$ of compact open subgroups of $\widehat{B}^\times$. We need the following fundamental result ([Vig80, Théorème III.4.3]).

**Theorem 1.3.1** (Strong approximation)**.** *Let $F$ be a number field and let $B$ be a quaternion algebra over $F$. Let $\mathbb{A}_\mathbb{Q}$ be the adèle ring over $\mathbb{Q}$ and let $S$ be a set of places of $F$ containing at least one place at which $B$ splits. Consider the group*

$$B_S^1 = \prod_{v \in S} B^1(F_v) \subseteq (B \otimes_\mathbb{Q} \mathbb{A}_\mathbb{Q})^1. \tag{1.8}$$

*Then $B^1 B_S^1$ is dense in $(B \otimes_\mathbb{Q} \mathbb{A}_\mathbb{Q})^1$.*

**Corollary 1.3.2.** *Let $F$ be a totally real number field.*

(i) *Let $B$ be an indefinite quaternion algebra over $F$. Then $B^1$ is dense in $\widehat{B}^1$.*
(ii) *Let $H$ be a definite quaternion algebra over $F$. Then for any finite prime $\mathfrak{p}$ at which $H$ is split, $H^1 H_\mathfrak{p}^1$ is dense in $\widehat{H}^1$.*

Let $K$ be a compact open subgroup of $\widehat{B}^\times$. Then we obtain subgroups $\mathrm{P}(K \cap B^+)$ and $\mathrm{P}(K \cap B^1)$ of the adjoint group $\mathrm{P}B^\times = B^\times/F^\times$ of $B^\times$.

**Proposition 1.3.3.** *Let $B$ be an indefinite algebra over a totally real number field $F$ and let $K' \subseteq K$ be two compact open subgroups of $\widehat{B}^\times$.*

(i) *We have*

$$[K \cap B^1 : K' \cap B^1] = \frac{[K : K']}{[\mathrm{nrd}(K) : \mathrm{nrd}(K')]}. \tag{1.9}$$

(ii) *We have*

$$[K \cap B^+ : K' \cap B^+] = \frac{h[K : K']}{[\mathrm{nrd}(K) : \mathrm{nrd}(K')]}, \tag{1.10}$$

*where* $h = |\mathrm{Im}(\mathrm{nrd}(K) \cap \mathbb{Z}_F^+ \longrightarrow \mathrm{nrd}(K)/\mathrm{nrd}(K'))|$.

(iii) *If* $K' \cap F^\times = K \cap F^\times$*, then the equalities in (i) and (ii) also hold for the indices* $[\mathrm{P}(K \cap B^1) : \mathrm{P}(K' \cap B)]$ *and* $[\mathrm{P}(K \cap B^+) : \mathrm{P}(K' \cap B^+)]$*, respectively.*

*Proof.* We prove (ii): case (i) is similar to case (ii), and (iii) is obvious. Consider the sequence of maps

$$(K \cap B^+)/(K' \cap B^+) \xrightarrow{\varphi} K/K' \xrightarrow{\psi} \mathrm{nrd}(K)/\mathrm{nrd}(K'). \tag{1.11}$$

The map $\varphi$ is injective. We prove (ii) by showing that $\mathrm{Im}(\varphi) = \psi^{-1}(N)$, where $N = \mathrm{Im}(\mathrm{nrd}(K) \cap \mathbb{Z}_F^+ \to \mathrm{nrd}(K)/\mathrm{nrd}(K'))$. The inclusion $\mathrm{Im}(\varphi) \subseteq \psi^{-1}(N)$ is trivial. Conversely, suppose we are given a coset $\widehat{k}K'$ mapping to an element $\widehat{n}$ of $N$ under $\psi$. Let $\widehat{k}' \in K'$ be such that $\mathrm{nrd}(\widehat{k}\widehat{k}'^{-1})$ is in $\mathrm{nrd}(K) \cap \mathbb{Z}_F^+$. Using Theorem 1.2.1(ii), we see that there exists an element $b$ of $K \cap B^+$ such that $\mathrm{nrd}(b) = \mathrm{nrd}(\widehat{k}\widehat{k}'^{-1})$. Therefore $\mathrm{nrd}(b^{-1}\widehat{k}\widehat{k}'^{-1}) = 1$. By Corollary 1.3.2(i), there exist a $b_1 \in B^1$ and a $k_1 \in K' \cap B^1$ for which $b_1 k_1 = b^{-1}\widehat{k}\widehat{k}'^{-1}$. But then $bb_1 = \widehat{k}\widehat{k}'^{-1}k_1^{-1}$ is in $K \cap B^+$ and represents the coset $\widehat{k}K'$. □

Since $\widehat{\mathcal{O}}^\times \cap F^\times = \mathcal{O}^\times \cap F^\times = \mathbb{Z}_F^\times$ for all orders $\mathcal{O}$ of $B$, we get:

**Corollary 1.3.4.** *Let* $n > 0$*. Let* $\mathcal{O}(\mathfrak{p}^n) \subset \mathcal{O}(1)$ *be an inclusion of a level* $\mathfrak{p}^n$ *Eichler order into a maximal order and let* $q = |\mathbb{Z}_F/\mathfrak{p}|$*. Then*

$$[\mathrm{P}\mathcal{O}(1)^1 : \mathrm{P}\mathcal{O}(\mathfrak{p}^n)^1] = [\mathrm{P}\mathcal{O}(1)^+ : \mathrm{P}\mathcal{O}(\mathfrak{p}^n)^+] = q^{n-1}(q+1). \tag{1.12}$$

**Corollary 1.3.5.** *Let* $\mathcal{O}(1)$ *be maximal and let* $\mathcal{O} \subseteq \mathcal{O}(1)$ *be a inclusion of orders such that* $\mathfrak{p}\mathcal{O}(1) \subset \mathcal{O}$*. Let* $\kappa$ *be the residue field* $\mathbb{Z}_F/\mathfrak{p}$*. Then the following statements hold.*

(i) *We have*

$$[\mathrm{P}\mathcal{O}(1)^1 : \mathrm{P}\mathcal{O}^1] = [\widehat{\mathcal{O}}(1)^\times : \widehat{\mathcal{O}}^\times]/d, \tag{1.13}$$

*where* $d \in \{1, 2\}$ *equals* 2 *if and only if* $\mathfrak{p}$ *is odd and the final column of the row corresponding to* $\mathcal{O}$ *in the table in Proposition 1.1.2 or 1.1.4 is given by* $U^{(1)}\mathbb{Z}_F^{\times 2}$.

(ii) *We have*

$$[\mathrm{P}\mathcal{O}(1)^+ : \mathrm{P}\mathcal{O}^+] = [\widehat{\mathcal{O}}(1)^\times : \widehat{\mathcal{O}}^\times]n/d, \tag{1.14}$$

*where* $d$ *is as above and* $n \in \{1, 2\}$ *equals* 2 *if and only if* $d$ *equals* 2 *and the canonical map* $\mathbb{Z}_F^+ \to \kappa^\times/\kappa^{\times 2}$ *is surjective.*

*Proof.* This follows from the fact that $\mathbb{Z}_{F,\mathfrak{p}}^\times/U_\mathfrak{p}^{(1)} \cong \kappa^\times$ is a cyclic group of order $|\mathbb{Z}_F/\mathfrak{p}| - 1$, which is odd if and only if $\mathfrak{p}$ is even. □

*Remark* 1.3.6. The corollaries above generalize to composite level. However, in our calculations in Section 4, the level will contain only one prime.

1.4. **Picard and type numbers.** Let $B$ be a quaternion algebra over a totally real number field $F$ and let $K \subset \widehat{B}^\times$ be a compact open subgroup. Then we can consider the *(right) Picard set*

$$\mathrm{Pic}_r(K) = B^\times \backslash \widehat{B}^\times / K. \tag{1.15}$$

Moreover, if $B$ satisfies (0.1), then we define the *narrow (right) Picard set*

$$\mathrm{Pic}_r(K\infty) = B^\times \backslash \{\pm 1\} \times \widehat{B}^\times / K. \tag{1.16}$$

Here $b \in B^\times$ acts on $\widehat{B}^\times$ through left multiplication and on $\{\pm 1\}$ through multiplication by the sign of $\mathrm{nrd}(b)$ at the split infinite place of $B$. We call the cardinality $|\mathrm{Pic}_r(K)|$ (resp. $|\mathrm{Pic}_r(K\infty)|$) the *Picard number* (resp. *narrow Picard number*) of $K$.

If $K = \widehat{\mathcal{O}}^\times$, where $\mathcal{O}$ is an order of $B$, then $\mathrm{Pic}_r(K)$ classifies the locally principal right $\mathcal{O}$-ideals up to left multiplication by elements of $B^\times$, *cf.* [Vig80, Section III.5.B]. Similarly, $\mathrm{Pic}_r(\mathcal{O}\infty)$ describes the set of equivalence classes of locally principal right $\mathcal{O}$-ideals equipped with an orientation at the split infinite place $\iota$ of $B$.

The *narrow type set* of $K$ is defined by

$$T(K\infty) = B^\times \backslash \{\pm 1\} \times \widehat{B}^\times / N(K) \tag{1.17}$$

Its cardinality is called the *narrow type number* of $K$. For $K = \widehat{\mathcal{O}}^\times$, it classifies the global conjugacy classes of the orders locally conjugate to $\mathcal{O}$ equipped with an orientation at $\iota$.

**Proposition 1.4.1.** *Consider $B$ and $K$ as above.*

(i) *The reduced norm map induces a bijection*

$$\mathrm{Pic}_r(K\infty) \xrightarrow{\sim} F_B^\times \backslash \{\pm 1\} \times \widehat{F}^\times / \mathrm{nrd}(K) \cong \mathrm{Cl}(K). \tag{1.18}$$

(ii) *Similarly, it induces a bijection*

$$T(K\infty) \xrightarrow{\sim} F_B^\times \backslash \{\pm 1\} \times \widehat{F}^\times / \mathrm{nrd}(N(K)) \cong \mathrm{Cl}(K). \tag{1.19}$$

*The cardinality $|T(K\infty)|$ is a power of 2.*

*Proof.* The isomorphisms (1.18) and (1.19) are generalizations of [Vig80, Corollaire III.5.7]. The final remark follows from the fact that $\widehat{F}^\times \subset N(K)$, which implies $\mathrm{nrd}(\widehat{F}^\times) = \widehat{F}^{\times 2} \subset \mathrm{nrd}(N(K))$. □

**Proposition 1.4.2.** *Let $B$ be as above and let $\mathcal{O}(\mathfrak{N})$ be a level $\mathfrak{N}$ Eichler order of $B$. Let $K = \widehat{\mathcal{O}}(\mathfrak{N})^\times$.*

(i) *Let $\mathfrak{p}$ be a prime of $F$ at which $B$ is split. Then*

$$\mathrm{nrd}(N(K_\mathfrak{p})) = \langle F_\mathfrak{p}^{\times 2} \mathbb{Z}_{F,\mathfrak{p}}^\times, \pi^{v_\mathfrak{p}(\mathfrak{N})} \rangle. \tag{1.20}$$

(ii) *Let $\mathfrak{p}$ be a prime of $F$ at which $B$ ramifies. Then* $\mathrm{nrd}(N(K_\mathfrak{p})) = F_\mathfrak{p}^\times$.

(iii) *Let* $\mathrm{Cl}[2] = \mathrm{Cl}(\infty)/2\mathrm{Cl}(\infty)$. *Then*

$$T(K\infty) \cong \mathrm{Cl}[2] / \langle \{\mathfrak{p} : \mathfrak{p} | \mathfrak{D}(B)^f\} \cup \{\mathfrak{p} : v_\mathfrak{p}(\mathfrak{N}) \text{ odd}\} \rangle. \tag{1.21}$$

*Proof.* Analogous to that of [Vig80, Corollaire III.5.7]. □

We refer to [KV10] for the determination of Picard and type numbers in definite quaternion algebras.

## 2. Canonical models

Throughout this section, let $F$ be a totally real number field, let $B$ be a quaternion algebra over $F$ satisfying (0.1) and let $K \subset \widehat{B}^\times$ be a compact open subgroup.

2.1. **Shimura curves.** Let $U$ be the Riemann surface $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$. The group $\mathrm{GL}_2(\mathbb{R})$ acts on $U$ through Möbius transformations, hence so does $B^\times$ via the embedding $\iota$ from Section 0.3. We can therefore construct the double quotient

$$Y(K) = B^\times \backslash U \times \widehat{B}^\times / K. \tag{2.1}$$

For an order $\mathcal{O}$ of $B$, we abbreviate $Y(\mathcal{O}) = Y(\widehat{\mathcal{O}}^\times)$.

**Proposition 2.1.1.** *Let $Y(K)$ be as in (2.1).*

(i) *Let $\pi_0(Y(K))$ be the set of connected components of $Y(K)$. Then*

$$\pi_0(Y(K)) \cong B^\times \backslash \{\pm 1\} \times \widehat{B}^\times / K = \mathrm{Pic}(K\infty) \tag{2.2}$$

*and the reduced norm map induces an isomorphism*

$$\pi_0(Y(K)) \xrightarrow{\sim} \mathrm{Cl}(K). \tag{2.3}$$

(ii) *Let $\{(a_i, \widehat{b}_i)\}$, with $a_i \in \{\pm 1\}$ and $\widehat{b}_i \in \widehat{B}^\times$, be a set of representatives for the quotient (2.2). Then there is an isomorphism*

$$Y(K) \cong \coprod_i X^{a_i}(\widehat{b}_i K \widehat{b}_i^{-1} \cap B^+). \tag{2.4}$$

*The quotient $Y(K)$ is compact if and only if $B$ is a division algebra.*

(iii) *There occur at most $|T(K\infty)|$ isomorphism classes of curves on the right hand side of (2.4).*

*Proof.* Part (i) follows from the identification $\pi_0(U) \cong \{\pm 1\}$. Part (ii) results from of Lemma 5.13 and Theorem 3.3 in [Mil]. As for part (iii), an element $\widehat{n}$ of $N(K)$ induces an automorphism of $Y(K)$ sending $[x, \widehat{b}]$ to $[x, \widehat{b}\widehat{n}]$. Under the isomorphism (2.3), the action of $n$ on $\pi_0(Y(K))$ is given by multiplication by $\mathrm{nrd}(\widehat{n})$. Components that are accordingly permuted are isomorphic. □

We let the *neutral component* $Y_0^+(K)$ of $Y(K)$ be the connected component of $Y(K)$ containing the class $[i, 1]$. There is an isomorphism $Y_0^+(K) \cong X^+(K \cap B^+)$. Here the group $K \cap B^+ \subset B^\times / F^\times$ is considered as a subgroup of $\mathrm{GL}_2(\mathbb{R})^+$ through $\iota$. We abbreviate $Y_0(K) = Y_0^+(K)$. Similarly, the connected component $Y_0^-(K)$ of $Y(K)$ containing $[-i, 1]$ is isomorphic to $X^-(K \cap B^+)$.

We call the automorphisms $[x, \widehat{b}] \mapsto [x, \widehat{b}\widehat{n}]$ of $Y(K)$ induced by elements $\widehat{n}$ of $N(K)$ *Atkin–Lehner automorphisms* of $Y(K)$. Let $\mathfrak{D} = \mathfrak{D}(B)^f$. Suppose $K = \widehat{\mathcal{O}}(\mathfrak{N})^\times$, where $\mathcal{O}(\mathfrak{N})$ is a level $\mathfrak{N}$ Eichler order of $B$. Let $\mathfrak{a}$ be a product of distinct primes at which $\mathfrak{D}\mathfrak{N}$ has odd valuation. Then by Proposition 1.4.2, there exists an element $\widehat{n}(\mathfrak{a})$ of $N(K)$ whose components in the quotient

$$N(K)/\widehat{F}^\times K \cong \prod_{\mathfrak{p}} N(K_\mathfrak{p})/F_\mathfrak{p}^\times K_\mathfrak{p} \cong \prod_{\mathfrak{p} | \mathfrak{D}\mathfrak{N}} \mathbb{Z}/2\mathbb{Z} \tag{2.5}$$

are non-trivial exactly at the primes dividing $\mathfrak{a}$. We denote the corresponding Atkin–Lehner involution by $a(\mathfrak{a})$.

**Theorem 2.1.2** ([Shi70]). *Let $Y(K)$ be as in (2.1).*

(i) *There exists a curve $\mathrm{Sh}(K)$ over $F$ that is a canonical model of $Y(K)$ over $F$. In particular $\mathrm{Sh}(K) \otimes_{F,\iota} \mathbb{C} \cong Y(K)$.*

(ii) *Let $K' \subseteq K$. Then the canonical map $Y(K') \to Y(K)$ is induced by an $F$-morphism $\mathrm{Sh}(K') \to \mathrm{Sh}(K)$.*

(iii) *Let $\widehat{b} \in \widehat{B}^\times$. Then the canonical isomorphism $Y(K) \to Y(\widehat{b}^{-1} K \widehat{b})$ given by $[x, \widehat{b}'] \mapsto [x, \widehat{b}'\widehat{b}]$ is induced by an $F$-morphism $\mathrm{Sh}(K) \to \mathrm{Sh}(\widehat{b}^{-1} K \widehat{b})$.*

*Proof.* Part (i) is [Car86, (1.1.1)]. Parts (ii) and (iii) follow from [Mil, Theorem 13.6]. □

We now turn to the arithmetic properties of the *Shimura curve* $\mathrm{Sh}(K)$. The scheme of connected components $\pi_0(\mathrm{Sh}(K))$ is a finite étale scheme over $\mathrm{Spec}(F)$ whose geometric points are given by

$$\pi_0(\mathrm{Sh}(K))(\overline{F}) = \pi_0(\mathrm{Sh}(K))(\mathbb{C}) = \pi_0(Y(K)). \tag{2.6}$$

This set of points inherits a left action of $G_F = \mathrm{Gal}(\overline{F}|F)$, which conversely determines $\pi_0(\mathrm{Sh}(K))$ as a scheme.

**Theorem 2.1.3** ([Car86], Section 1.2)**.** *Under the isomorphism (2.3), the action $\sigma \in G_F$ on $\mathrm{Cl}(K)$ is given by $\sigma([x]) = [\widehat{s}x]$, where $\widehat{s} \in \widehat{F}^\times$ is any finite idèle whose image under the Artin reciprocity map $\widehat{F}^\times \hookrightarrow \mathbb{A}_F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}|F)$ equals $\sigma|_{F^{\mathrm{ab}}}$.*

Therefore $\pi_0(\mathrm{Sh}(K)) \cong \mathrm{Spec}(F_K)$. Hence there exist models $\mathrm{Sh}_0^{\pm}(K)$ over $F_K$ for the components $Y_0^{\pm}(K)$. Moreover, $\mathrm{Sh}(K)$ is isomorphic to the scheme

$$\mathrm{Sh}_0(K) \longrightarrow \mathrm{Spec}(F_K) \longrightarrow \mathrm{Spec}(F) \tag{2.7}$$

over $F$. In other words, there is a finite decomposition

$$(\mathrm{Sh}(K))_{F_K} \cong \coprod_{\sigma \in \mathrm{Gal}(F_K|F)} \mathrm{Sh}_0(K)^\sigma \tag{2.8}$$

over $F_K$. We also call the models $\mathrm{Sh}_0^{\pm}(K)$ of $Y_0^{\pm}(K)$ over $F_K$ canonical (*cf.* [Shi70]).

The Jacobian $J(K) = \mathrm{Jac}(\mathrm{Sh}(K))$ is an abelian variety over $F$. As in the discussion before Theorem B in [Zha01], we see that if we let $J_0^{\pm}(K) = \mathrm{Jac}(\mathrm{Sh}_0^{\pm}(K))$, then $J_0(K)$ is an abelian variety over $F_K$ for which

$$J(K) \cong \mathrm{Res}_{F_K|F}(J_0(K)). \tag{2.9}$$

Here Res denotes Weil restriction of scalars.

**Proposition 2.1.4.** *Let $n \in N(K)$. Then the Atkin–Lehner automorphism of $Y(K)$ associated to $n$ descends to an $F$-automorphism of $\mathrm{Sh}(K)$. In particular, there occur at most $|T(K\infty)|$ isomorphism classes of curves over $F_K$ on the right hand side of (2.8).*

*Proof.* Considering Theorem 2.1.2(iii), this follows from Proposition 2.1.1(iii). □

**Proposition 2.1.5.** *Let $\sigma \in G_F$. Then the Jacobians of the curves $\mathrm{Sh}_0(K)^\sigma$ and $\mathrm{Sh}_0(K)$ are isogenous over $F_K$.*

*Proof.* By weak approximation for $F^\times$ and surjectivity of the map $\mathrm{nrd} : K_\mathfrak{p} \to \widehat{\mathbb{Z}}_{F,\mathfrak{p}}^\times$ at primes $\mathfrak{p}$ where $K$ is maximal (*cf.* Theorem 1.2.1(iii)), we see that we can choose the representatives $(a_i, \widehat{b}_i)$ in Proposition 2.1.1(ii) to satisfy

$$a_i = 1 \text{ and } \widehat{b}_{i,\mathfrak{p}} = 1 \text{ at } \mathfrak{p} \text{ where } K \text{ is not maximal or } B \text{ ramifies.} \tag{2.10}$$

Fix $i$. Then $K' = K \cap \widehat{b}_i K \widehat{b}_i^{-1}$ is a compact open subgroup of $\widehat{B}^\times$ and therefore of finite index in both $K$ and $\widehat{b}_i K \widehat{b}_i^{-1}$. This gives rise to a correspondence

$$\mathrm{Sh}(K) \longleftarrow \mathrm{Sh}(K') \longrightarrow \mathrm{Sh}(\widehat{b}_i K \widehat{b}_i^{-1}). \tag{2.11}$$

We claim that the correspondence (2.11) induces isomorphisms on connected components. Indeed, by construction, at primes $\mathfrak{p}$ where $\widehat{b}_i$ is non-trivial, $K'_\mathfrak{p}$ is given by the unit group of an Eichler order, hence $\mathrm{nrd}(K'_\mathfrak{p}) = \widehat{\mathbb{Z}}_{F,\mathfrak{p}}^\times = \mathrm{nrd}(K_\mathfrak{p})$. At $\mathfrak{p}$ where $\widehat{b}_i$ is trivial, obviously $\mathrm{nrd}(K_\mathfrak{p}) = \mathrm{nrd}(K'_\mathfrak{p})$ since $K_\mathfrak{p} = K'_\mathfrak{p}$. Hence $\mathrm{nrd}(K') = \mathrm{nrd}(K)$, which proves the claim in light of Proposition 2.1.1(i).

The correspondence in (2.11) induces trivial maps on the schemes of connected components by Theorem 2.1.2(ii). Consequently, it induces a correspondence over $F_K$ between the neutral components $\mathrm{Sh}_0(K)$ and $\mathrm{Sh}_0(\widehat{b}_i K \widehat{b}_i^{-1})$. This is the same as giving an isogeny of the corresponding Jacobians. Since the neutral component of

$\mathrm{Sh}(\widehat{b}_i K \widehat{b}_i^{-1})$ can be identified with the connected component of $\mathrm{Sh}(K)$ containing $(a_i, \widehat{b}_i)$, the Proposition follows from (2.8). $\square$

**Theorem 2.1.6.** *Let $\mathfrak{p}$ be a prime of $F$, and suppose that $K$ is of the form $K = K_{\mathfrak{p}} \times K^{\mathfrak{p}}$, where $K_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ and $K^{\mathfrak{p}} \subset \widehat{B}^{\mathfrak{p}}$. Let $\mathfrak{C}$ be the conductor of $J(K)$ as an abelian variety. Then $v_{\mathfrak{p}}(\mathfrak{C})$ only depends on $K_{\mathfrak{p}}$. Furthermore,*

(i) *If $\mathfrak{p} \nmid \mathfrak{D}(B)$ and $K_{\mathfrak{p}}$ is maximal at $\mathfrak{p}$, then $v_{\mathfrak{p}}(\mathfrak{C}) = 0$.*
(ii) *If $\mathfrak{p} \nmid \mathfrak{D}(B)$ and $K_{\mathfrak{p}}$ is the unit group of a level $\mathfrak{N}$ Eichler order of $B_{\mathfrak{p}}$, then $v_{\mathfrak{p}}(\mathfrak{N}) = 1$ implies $v_{\mathfrak{p}}(\mathfrak{C}) = 1$.*
(iii) *If $\mathfrak{p} \mid \mathfrak{D}(B)$ and $K_{\mathfrak{p}}$ is maximal at $\mathfrak{p}$, then $v_{\mathfrak{p}}(\mathfrak{C}) = 1$.*

*Proof.* The first two statements follow from [Car86]. The third is a consequence of the $\mathfrak{p}$-adic uniformization of $\mathrm{Sh}(K)$ (*cf*. Proposition 3.1.14(ii)). $\square$

Finally, we give an adèlic version of the main result in [DN67], which relates Shimura curves coming from different quaternion algebras.

**Theorem 2.1.7.** *Let $B$ be a quaternion algebra over $F$ that is split at a unique infinite place $\iota$. Let $B'$ be an algebra that is ramified at the same finite places as $B$ and that is split at a unique infinite place $\iota'$. Let*

$$\widehat{B} = \prod_{\mathfrak{p}}{}' B_{\mathfrak{p}} \xrightarrow{\varphi} \prod_{\mathfrak{p}}{}' B'_{\mathfrak{p}} = \widehat{B}' \tag{2.12}$$

*be an isomorphism of restricted direct products. Let $K$ be a compact open subgroup of $\widehat{B}^{\times}$. Then if we let $K' = \varphi(K)$, there exists an isomorphism*

$$\mathrm{Sh}(K) \otimes_{F,\iota'} \mathbb{C} \cong \mathrm{Sh}(K') \otimes_{F,\iota'} \mathbb{C} \tag{2.13}$$

*of curves over $\mathbb{C}$.*

Let $\sigma$ be an automorphism of $F$. Then if the $F$-algebra structure of $B$ is given by $F \xrightarrow{i} B$, we can consider the $F$-algebra ${}^{\sigma}B$ obtained by

$$F \xrightarrow{\sigma^{-1}} F \xrightarrow{i} B. \tag{2.14}$$

The identity map on $B$ is an isomorphism of $\mathbb{Q}$-algebras $B \to {}^{\sigma}B$. We have $\mathfrak{D}({}^{\sigma}B) = \sigma(\mathfrak{D}(B))$. Here, for a place $v$ of $F$, we denote by $\sigma v$ the place $v \circ \sigma^{-1}$.

Upon tensoring, we obtain an isomorphism of $\mathbb{Q}$-algebras

$$\widehat{B} = B \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}^{f} \longrightarrow {}^{\sigma}B \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}^{f} = {}^{\sigma}\widehat{B} \tag{2.15}$$

Let $K$ be a compact open subgroup of $\widehat{B}^{\times}$ and let ${}^{\sigma}K$ be equal to $K$, but this time considered as a subgroup of ${}^{\sigma}\widehat{B}^{\times}$ through the isomorphism (2.15). Then clearly there is an isomorphism

$$Y(K) = B^{\times} \backslash U \times \widehat{B}^{\times} / K \cong {}^{\sigma}B^{\times} \backslash U \times {}^{\sigma}\widehat{B}^{\times} / {}^{\sigma}K = Y({}^{\sigma}K), \tag{2.16}$$

therefore

$$\mathrm{Sh}(K) \otimes_{F,\iota} \mathbb{C} \cong \mathrm{Sh}({}^{\sigma}K) \otimes_{F,\sigma\iota} \mathbb{C}. \tag{2.17}$$

Combining the isomorphisms (2.13) and (2.17), one can show that $Y(K)$ is defined over a proper subfield of $F$ in a large variety of cases, such as the following.

**Corollary 2.1.8.** *Let $F$ be Galois over $\mathbb{Q}$ and let $B$ be an algebra over $F$ whose finite discriminant $\mathfrak{D}(B)^f$ is $\mathrm{Gal}(F|\mathbb{Q})$-invariant. Let $K = \widehat{\mathcal{O}}^{\times}$, where $\mathcal{O}$ is an Eichler order of $\mathrm{Gal}(F|\mathbb{Q})$-invariant level $\mathfrak{N}$. Then the field of moduli of $Y(K)$ equals $\mathbb{Q}$.*

*Proof.* We have to show that

$$\mathrm{Sh}(K)\otimes_{F,\iota}\mathbb{C}\cong\mathrm{Sh}(K)\otimes_{F,\iota'}\mathbb{C} \tag{2.18}$$

for all pairs of real places $\iota$ and $\iota'$ of $F$. Given $\iota$ and $\iota'$, there is an automorphism $\sigma$ of $F$ such that $\iota'=\sigma\iota$. Then by (2.17), we have

$$\mathrm{Sh}(K)\otimes_{F,\iota}\mathbb{C}\cong\mathrm{Sh}({}^{\sigma}K)\otimes_{F,\sigma\iota}\mathbb{C}=\mathrm{Sh}({}^{\sigma}K)\otimes_{F,\iota'}\mathbb{C}. \tag{2.19}$$

Note that ${}^{\sigma}K$ comes from a level $\sigma(\mathfrak{N})=\mathfrak{N}$ Eichler order of ${}^{\sigma}B$. By the Galois invariance of $\mathfrak{D}(B)^f$ we can take $B'={}^{\sigma}B$ in Theorem 2.1.7. We let $K'={}^{\sigma}K$. Then since both $K'$ and $K$ come from level $\mathfrak{N}$ Eichler orders, we can choose the isomorphism in (2.12) such that $K$ is mapped to $K'$. Therefore (2.13) yields

$$\mathrm{Sh}({}^{\sigma}K)\otimes_{F,\iota'}\mathbb{C}=\mathrm{Sh}(K')\otimes_{F,\iota'}\mathbb{C}\cong\mathrm{Sh}(K)\otimes_{F,\iota'}\mathbb{C}. \tag{2.20}$$

Combining (2.19) and (2.20), we obtain (2.18). □

*Remark* 2.1.9. See [Hal09, Proposition 1] for a related result. In Section 4, we will see that the hypotheses of Corollary 2.1.8 do not imply that the canonical model $\mathrm{Sh}(K)$ descends to $\mathbb{Q}$.

2.2. **From $K\cap B^+$ to $K\cap B^1$.** This section considers the subgroup $K\cap B^1$ of $K\cap B^+$. In the previous section, we obtained the curve $X(K\cap B^+)$ as the neutral component of the Riemann surface $Y(K)$, resulting in a canonical model $\mathrm{Sh}_0(K)$ for this curve over $F_K$. For reasons that will become clear in the next section, we are also interested in obtaining a canonical model of the curve $X(K\cap B^1)$. To obtain this curve as a Shimura curve, this section constructs compact open groups $K'\subset\widehat{B}^\times$ that are slightly smaller than $K$.

**Proposition 2.2.1.** *The group $\mathrm{P}(K\cap B^1)$ is a normal subgroup of $\mathrm{P}(K\cap B^+)$ of finite index. Let $N=\mathrm{nrd}(K)$. Then we have an isomorphism*

$$\mathrm{P}(K\cap B^+)/\mathrm{P}(K\cap B^1)\cong(N\cap\mathbb{Z}_F^+)/(N\cap\mathbb{Z}_F^{\times 2}). \tag{2.21}$$

*The latter group is isomorphic to a subgroup of $\mathrm{Ker}(\mathrm{Cl}(\infty)\to\mathrm{Cl}(1))$, with equality holding if $K=\widehat{\mathcal{O}}^\times$ for an Eichler order $\mathcal{O}$.*

*Proof.* Let $(K\cap B^+)^{(2)}$ be the subgroup of $K\cap B^+$ consisting of those elements whose norm is in $\mathbb{Z}_F^{\times 2}$. Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}_F^\times & \longrightarrow & (K\cap B^+)^{(2)}\mathbb{Z}_F^\times & \longrightarrow & \mathrm{P}(K\cap B^1) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}_F^\times & \longrightarrow & K\cap B^+\mathbb{Z}_F^\times & \longrightarrow & \mathrm{P}(K\cap B^+) & \longrightarrow & 1 \end{array} \tag{2.22}$$

Considering this diagram, we see that the canonical map

$$(K\cap B^+)\mathbb{Z}_F^\times/(K\cap B^+)^{(2)}\mathbb{Z}_F^\times\longrightarrow\mathrm{P}(K\cap B^+)/\mathrm{P}(K\cap B^1) \tag{2.23}$$

is an isomorphism. On the other hand, by Theorem 1.2.1(ii), the reduced norm map induces an isomorphism

$$\begin{aligned}(K\cap B^+)\mathbb{Z}_F^\times/(K\cap B^+)^{(2)}\mathbb{Z}_F^\times &\longrightarrow(N\cap\mathbb{Z}_F^+)\mathbb{Z}_F^{\times 2}/(N\cap\mathbb{Z}_F^{\times 2})\mathbb{Z}_F^{\times 2}\\ &\cong(N\cap\mathbb{Z}_F^+)/(N\cap\mathbb{Z}_F^{\times 2}).\end{aligned} \tag{2.24}$$

As for the second part, there is a canonical map

$$\mathbb{Z}_F^\times/\mathbb{Z}_F^{\times 2}\longrightarrow\mathbb{A}_F^{\infty\times}/(\mathbb{A}_F^{\infty\times})^2=\prod_{v|\infty}U_v^{(0)}/U_v^{(1)}. \tag{2.25}$$

Here $U_v^{(0)} = F_v^\times \cong \mathbb{R}^\times$, and $U_v^{(1)}$ is the connected component of $U_v^{(0)}$ containing 1.

The kernel of the quotient map $\mathrm{Cl}(\infty) \to \mathrm{Cl}(1)$ is isomorphic to the cokernel of the map (2.25) (*cf.* the proof of Lemma 2.2.2). The kernel of (2.25) is given by $\mathbb{Z}_F^+/\mathbb{Z}_F^{\times 2}$. Now because of Dirichlet's unit theorem and the fact that any number field has a root of unity of order 2, both of the groups in (2.25) have the same cardinality, to wit $2^{\deg(F|\mathbb{Q})}$. Therefore the kernel and cokernel of (2.25) have the same cardinality. Since both have exponent 2, these groups are isomorphic. To conclude the proof, note that $\mathrm{nrd}(\widehat{\mathcal{O}}^\times) = \widehat{\mathbb{Z}}_F^\times$ for all Eichler orders $\mathcal{O}$. □

Consequently, if the narrow class group of $F$ is non-trivial, then usually $X(K \cap B^1)$ will be a non-trivial cover of $X(K \cap B^+)$. Working directly with the algebraic group $\mathrm{Res}_{F|\mathbb{Q}}(B^1)$ to deal with $K \cap B^1$ is unfortunately out of the question, as there is no Shimura datum for $\mathrm{Res}_{F|\mathbb{Q}}(B^1)$. However, one can find a way around the problem by considering suitable compact open subgroups of $K$ of small index. The key point is the following lemma:

**Lemma 2.2.2.** *Let $F$ be a number field and let $N$ be a compact open subgroup of $\widehat{F}^\times$. Then there exists a compact open subgroup $N'$ of $N$ satisfying the following properties:*

(i) *The canonical quotient map* $\mathrm{Cl}(N'\infty) \to \mathrm{Cl}(N\infty)$ *is an isomorphism.*
(ii) $\mathbb{Z}_F^+ \cap N' = \mathbb{Z}_F^{\times 2} \cap N$.

*Moreover, for any such subgroup $N'$, the canonical map*

$$(\mathbb{Z}_F^+ \cap N)/(\mathbb{Z}_F^+ \cap N') \longrightarrow N/N' \tag{2.26}$$

*is an isomorphism.*

*Proof.* The Čebotarev density theorem implies that given a nonsquare $x$ in $F$, the set of primes of $F$ at which $x$ is a square has density $1/2$. This allows us to construct $N'$ through a repeated shrinking process.

We may suppose that the inclusion $\mathbb{Z}_F^{\times 2} \cap N \subseteq \mathbb{Z}_F^+ \cap N$ is strict: otherwise we can take $N' = N$. Let $x$ be a non-square element of $\mathbb{Z}_F^+ \cap N$ and let $\mathfrak{p}$ be a prime such that $x$ is not a square at $\mathfrak{p}$ and such that $N_\mathfrak{p}$ does not equal $\mathbb{Z}_{F,\mathfrak{p}}^{\times 2}$. Such a $\mathfrak{p}$ exists: for example, one can take $\mathfrak{p}$ to be an odd prime at which $x$ is not a square and where $N$ is maximal.

Let $N'_\mathfrak{p}$ be an index 2 subgroup of $N_\mathfrak{p}$ such that $N'_\mathfrak{p}$ contains $\mathbb{Z}_{F,\mathfrak{p}}^{\times 2}$ and such that $x$ is not in $N'_\mathfrak{p}$. Construct $N' = N'_\mathfrak{p} \times \prod_{\mathfrak{q} \neq \mathfrak{p}} N_\mathfrak{q}$. Now $x$ is not in $N'$; on the other hand, since $N'_\mathfrak{p}$ contains $\mathbb{Z}_{F,\mathfrak{p}}^{\times 2}$, one still has that $\mathbb{Z}_F^{\times 2} \cap N$ is contained in $\mathbb{Z}_F^+ \cap N'$. By construction, we also have $[\mathbb{Z}_F^+ \cap N : \mathbb{Z}_F^{\times 2} \cap N] = 2[\mathbb{Z}_F^+ \cap N' : \mathbb{Z}_F^{\times 2} \cap N]$.

We now check that the quotient map in (i) is an isomorphism for $N$ and $N'$. Its kernel is given by

$$\frac{F^+ N}{F^+ N'} = \frac{F^+ N' N_\mathfrak{p}}{F^+ N'} \cong \frac{N_\mathfrak{p}}{N_\mathfrak{p} \cap F^+ N'} \tag{2.27}$$

$$\cong \frac{N_\mathfrak{p}}{(\mathbb{Z}_F^+ \cap N_\mathfrak{p}) N'_\mathfrak{p}}. \tag{2.28}$$

In this string of isomorphisms, we have embedded $F^\times$ diagonally in the idéles $\widehat{F}^\times$ in (2.27), while we embedded it in the factor $F_\mathfrak{p}^\times$ in (2.28). The group $(\mathbb{Z}_F^+ \cap N_\mathfrak{p})N'_\mathfrak{p}$ contains $N'_\mathfrak{p}$, hence is at worst of index 2 in $N_\mathfrak{p}$. On the other hand, it also contains $x \in \mathbb{Z}_F^+ \cap N_\mathfrak{p}$, which is not in $N'_\mathfrak{p}$, so in fact it equals $N_\mathfrak{p}$. Hence the quotient map $\mathrm{Cl}(N'\infty) \to \mathrm{Cl}(N\infty)$ is indeed an isomorphism. Inductively repeating this procedure above, one obtains a $N'$ as in the lemma, since $\mathbb{Z}_F^+/\mathbb{Z}_F^{\times 2}$ is finitely generated.

Now for the last statement of the lemma. By (ii), the map (2.26) is injective: it remains to prove that $(\mathbb{Z}_F^+ \cap N)N' = N$. By (i), the quotient $F^+N/F^+N' = F^+N'N/F^+N' \cong N/(F^+N' \cap N)$ is trivial, hence we can conclude by noting that $(\mathbb{Z}_F^+ \cap N)N' = \mathbb{Z}_F^+ N' \cap N = F^+ N' \cap N$. □

Consider a compact open subgroup $K$ of $\widehat{B}^\times$. By applying the reduced norm map, one obtains a compact open subgroup $N = \mathrm{nrd}(K)$ of $\widehat{F}^\times$. Choosing an $N'$ satisfying the properties in the lemma, we can then construct the subgroup

$$K' = K \cap \mathrm{nrd}^{-1}(N') \tag{2.29}$$

which is again compact open because of the continuity of the reduced norm map. By construction of $K'$, we have achieved our objective:

**Proposition 2.2.3.** *Let $K$, $N'$ and $K'$ be as above.*

(i) *We have $Y_0^\pm(K') \cong X^\pm(K \cap B^1)$. Similarly, the other components of $Y(K')$ do not depend on the choice of $N'$.*
(ii) *The canonical map $\mathrm{Sh}(K') \to \mathrm{Sh}(K)$ induces an $F$-isomorphism $\pi_0(\mathrm{Sh}(K')) \to \pi_0(\mathrm{Sh}(K))$. In particular, $F_{K'} = F_K$, and the canonical map $Y_0^\pm(K') \to Y_0^\pm(K)$ descends to an $F_K$-morphism $\mathrm{Sh}_0^\pm(K') \to \mathrm{Sh}_0^\pm(K)$.*

*Remark* 2.2.4. As we shall see explicitly in Section 4, different choices for $K'$ can give rise to different canonical models of the same Riemann surface. To avoid confusion, we therefore call $\mathrm{Sh}_{(0)}^\pm(K')$ a *$K'$-model* of the Riemann surface $Y_{(0)}^\pm(K')$.

## 2.3. **Arithmetic $(1;e)$-curves.**

**Definition 2.3.1.** *An* arithmetic subgroup *of* $\mathrm{PGL}_2(\mathbb{R})^+$ *is a subgroup that is commensurable with a group $\mathrm{P}(K \cap B^+)$ for some choice of $F$, $B$, $\iota$ and $K$.*

In [Tak83], Takeuchi determined the 71 $\mathrm{PGL}_2(\mathbb{R})$-conjugacy classes of arithmetic $(1;e)$-groups. A key ingredient for his classification is the following result ([Tak83, Theorem 3.4]):

**Lemma 2.3.2.** *Let $\Gamma$ be an arithmetic $(1;e)$-group. Then $\Gamma^{(2)} = \langle \gamma^2 \mid \gamma \in \Gamma \rangle$ is a normal subgroup of $\Gamma$. The curve $X^+(\Gamma)$ (resp. $X^-(\Gamma)$) is isomorphic to $X^+(\Gamma^{(2)})$ (resp. $X^-(\Gamma^{(2)})$). Choosing such an isomorphism, the canonical maps $X^\pm(\Gamma^{(2)}) \to X^\pm(\Gamma)$ induce maps on Jacobians that are isomorphic to multiplication by* 2.

*Let $\widetilde{\Gamma}$ be the lift of $\Gamma$ to $\mathrm{SL}_2(\mathbb{R})$ defined in* [Tak83, Section 3]. *The field $F = \mathbb{Q}(\mathrm{tr}(\widetilde{\Gamma}^{(2)}))$ is a totally real subfield of $\mathbb{C}$. The algebra $B = F[\widetilde{\Gamma}^{(2)}]$ is a quaternion algebra over $F$ that satisfies (0.1). The ring $\mathbb{Z}_F[\widetilde{\Gamma}^{(2)}]$ is an order of $B$.*

We abuse notation by writing $\mathbb{Z}_F[\Gamma^{(2)}]$ for $\mathbb{Z}_F[\widetilde{\Gamma}^{(2)}]$. Lemma 2.3.2 gives rise to the correspondence

$$X^\pm(\Gamma) \overset{''[2]''}{\longleftarrow} X^\pm(\Gamma^{(2)}) \longrightarrow X^\pm(\mathbb{Z}_F[\Gamma^{(2)}]^1). \tag{2.30}$$

For a Fuchsian group $\Gamma$, we denote the Jacobian of $X^\pm(\Gamma)$ by $J^\pm(\Gamma)$. A model $C$ of $X^\pm(\Gamma)$ over a field $L$ will give rise to an elliptic curve $\mathrm{Jac}(C)$ that is a model of $J^\pm(\Gamma)$ over $L$.

**Definition 2.3.3.** *Let $\Gamma$ be a Fuchsian group. Suppose that for some compact open $K$ we have*

$$\mathrm{P}(K \cap B^+) \subseteq \mathrm{P}\Gamma \subseteq \mathrm{P}(N(K) \cap B^+). \tag{2.31}$$

*Then we call the Atkin–Lehner quotient of $\mathrm{Sh}_0^\pm(K)$ corresponding to $X(\Gamma)$ a* canonical model *of $X^\pm(\Gamma)$ over $F_K$. The corresponding quotient of the Jacobian $J_0^\pm(K)$ is called a* canonical model *of $J^\pm(\Gamma)$.*

*Remark* 2.3.4. If $\Gamma$ is a $(1;e)$-group, then the canonical models of the curve $X^{\pm}(\Gamma)$ and the abelian variety $J^{\pm}(\Gamma)$ are isomorphic as curves, since the point on the Atkin–Lehner quotient of $\mathrm{Sh}_0^{\pm}(K)$ corresponding to the elliptic point of $X^{\pm}(\Gamma)$ will be rational. Indeed, as in [Ogg83], the elliptic points of $\mathrm{Sh}(K)$ are given by a union of orbits of CM-points. Since these orbits are defined over $F$, the elliptic points of $\mathrm{Sh}_0^{\pm}(K)$, being unique, are defined over the base field $F_K$ of $\mathrm{Sh}_0^{\pm}(K)$.

Our strategy for finding canonical models for $X^{\pm}(\Gamma)$ for arithmetic $(1;e)$-groups $\Gamma$ is the following. Given $\Gamma$, we consider the order $\mathcal{O} = \mathbb{Z}_F[\Gamma^{(2)}]$ and let $K_0 = \widehat{\mathcal{O}}^{\times}$. Then $Y_0^{\pm}(K_0) = X^{\pm}(\mathbb{Z}_F[\Gamma^{(2)}]^{+})$, which allows a cover by $X^{\pm}(\Gamma^{(2)})$. Motivated by the definition above, our goal is to find a group $K$ such that the indices $[K : K \cap K_0]$ and $[K_0 : K \cap K_0]$ are both small and such that $K$ satisfies (2.31); as we shall see in Section 4, $K_0$ itself need not satisfy these demands. The following lemma is of great use for finding a suitable $K$. It is proved in the same way as [Sij11, Lemma 1.2].

**Lemma 2.3.5.** *Let $\mathcal{O}$ be an order of $B$ satisfying $\mathcal{O} = \mathbb{Z}_F[\mathcal{O}^1]$ and let $K = \widehat{\mathcal{O}}^{\times}$. Then*

$$N_{\mathrm{PGL}_2(\mathbb{R})}(\mathrm{P}(K \cap B^1)) = N_{\mathrm{PGL}_2(\mathbb{R})}(\mathrm{P}(K \cap B^{+})) = \mathrm{P}(N_{\widehat{B}^{\times}}(K) \cap B^{+}). \tag{2.32}$$

*Remark* 2.3.6. We will see examples of groups $\Gamma$ with $\Gamma^{(2)} \subsetneq \mathbb{Z}_F[\Gamma^{(2)}]^1$ and orders $\mathcal{O}$ with $\mathbb{Z}_F[\mathcal{O}^1] \subsetneq \mathcal{O}$ in the calculations in Section 4. However, in no case were the corresponding normalizers different.

In Section 4, we usually take $K$ to equal $\widehat{\mathcal{O}}^{\times} \cap \mathrm{nrd}^{-1}(N')$, where $\mathcal{O}$ is an order containing $\mathbb{Z}_F[\Gamma^{(2)}]$ and satisfying $\mathcal{O} = \mathbb{Z}_F[\mathcal{O}^1]$, and where $N'$ is chosen as in Lemma 2.2.2. Then frequently

$$\Gamma^{(2)} \subseteq \mathrm{P}(K \cap B^{+}) \subseteq \Gamma. \tag{2.33}$$

In such a case, Lemma 2.3.2 shows that $\mathrm{P}\Gamma \subseteq N(\mathrm{P}(K \cap B^{+}))$, whence $\mathrm{P}(K \cap B^{+}) \subseteq \mathrm{P}\Gamma \subset \mathrm{P}(N(K) \cap B^{+})$ by Lemma 2.3.5, giving rise to canonical models of $X^{\pm}(\Gamma)$ as in Definition 2.3.3.

## 3. Uniformization

Let $B$ be a quaternion algebra satisfying (0.1) that ramifies at a prime $\mathfrak{p}$ of $F$. Let $K$ be a compact open subgroup of $\widehat{B}^{\times}$ decomposing as $K_{\mathfrak{p}} \times K^{\mathfrak{p}}$, where $K_{\mathfrak{p}}$ is the maximal compact subgroup of $B_{\mathfrak{p}}^{\times}$. This section explores the consequences of the $\mathfrak{p}$-adic uniformization of $\mathrm{Sh}(K)$ constructed by Boutot–Zink in [BZ] and by Varshavsky in [Var98]. These uniformizations generalize the results obtained by Čerednik–Drinfel'd over $\mathbb{Q}$ that were used in [Kur79] and [GR06].

Throughout this section, we use the notion of a graph from [Kur79, Definition 3-1]. In particular, we allow graphs to have oriented edges equal to their own inverse.

3.1. **Dual graphs.** Let $F_{\mathfrak{p}}^{\mathrm{unr}}$ be the maximal unramified extension of $F_{\mathfrak{p}}$, with ring of integers $\mathbb{Z}_{F,\mathfrak{p}}^{\mathrm{unr}}$. Let $\pi$ be a uniformizer of $\mathbb{Z}_{F,\mathfrak{p}}^{\mathrm{unr}}$. Given a scheme $X$ over $\mathbb{Z}_{F,\mathfrak{p}}$, we let $\widehat{X}$ denote the completion of $X$ along its special fiber. For a $\mathbb{Z}_{F,\mathfrak{p}}$-algebra $R$, the formal spectrum of $R$ is denoted by $\mathrm{Spf}(R)$, and we let

$$\widehat{X}^{\mathrm{unr}} = \widehat{X} \times_{\mathrm{Spf}(\mathbb{Z}_{F,\mathfrak{p}})} \mathrm{Spf}(\mathbb{Z}_{F,\mathfrak{p}}^{\mathrm{unr}}). \tag{3.1}$$

Let $\widehat{\Omega}_{\mathfrak{p}}$ be the $\mathfrak{p}$-adic upper half-plane. This is a formal scheme over $\mathbb{Z}_{F,\mathfrak{p}}$ that represents the functor defined in [BC91, Section 1.5]. The scheme $\widehat{\Omega}_{\mathfrak{p}}$ admits a natural action by the group $\mathrm{GL}_2(F_{\mathfrak{p}})$, which factorizes through the adjoint group $\mathrm{PGL}_2(F_{\mathfrak{p}})$. The special fiber of $\widehat{\Omega}_{\mathfrak{p}}$ consists of a tree of rational curves. As a graph with an action of $\mathrm{GL}_2(F_\mathfrak{p})$, the dual graph of this special fiber is isomorphic to the $\mathfrak{p}$-adic Bruhat-Tits tree, which we denote by $T_\mathfrak{p}$. We denote

$$\widehat{\Omega}_\mathfrak{p}^{\mathrm{unr}} = \widehat{\Omega}_\mathfrak{p} \times_{\mathrm{Spf}(\mathbb{Z}_{F,\mathfrak{p}})} \mathrm{Spf}(\mathbb{Z}_{F,\mathfrak{p}}^{\mathrm{unr}}). \tag{3.2}$$

Let $H$ be a quaternion algebra over $F$ of discriminant $\mathfrak{D}(B)^f\infty/\mathfrak{p}$. That is to say, $H$ is ramified everywhere at infinity, and its ramification behavior at the non-archimedean places is exactly the same as that of $B$, except at $\mathfrak{p}$, where $B$ is ramified and $H$ splits. Choosing an isomorphism of restricted direct products

$$\widehat{H}^\mathfrak{p} = \prod_{\mathfrak{q}:\mathfrak{q}\neq\mathfrak{p}}{}' H_\mathfrak{q} \cong \prod_{\mathfrak{q}:\mathfrak{q}\neq\mathfrak{p}}{}' B_\mathfrak{q} = \widehat{B}^\mathfrak{p}, \tag{3.3}$$

we obtain a left action of $\widehat{H}^{\mathfrak{p}\times}$ on $\widehat{B}^{\mathfrak{p}\times}$.

Since $K_\mathfrak{p}$ is the maximal compact open subgroup of $B_\mathfrak{p}^\times$, the norm map induces an isomorphism $B_\mathfrak{p}^\times/K_\mathfrak{p} \xrightarrow{\sim} F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times$, *cf.* [Vig80, Lemme II.1.5]. The group $H_\mathfrak{p}^\times$ acts on $F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times$ through its own reduced norm map $H_\mathfrak{p}^\times \to F_\mathfrak{p}^\times$. We obtain a corresponding action of $H_\mathfrak{p}^\times$ on $B_\mathfrak{p}^\times/K_\mathfrak{p}$.

Combining the two actions above, we obtain an action of $\widehat{H}^\times = H_\mathfrak{p}^\times \times \widehat{H}^{\mathfrak{p}\times}$ on the quotient $\widehat{B}^\times/K$, whence an induced action of $H^\times \subset \widehat{H}^\times$. We can also make the group $H^\times$ act on $\widehat{\Omega}_\mathfrak{p}$, and hence on $\widehat{\Omega}_\mathfrak{p}^{\mathrm{unr}}$, after choosing an isomorphism of groups $H_\mathfrak{p}^\times \cong \mathrm{GL}_2(F_\mathfrak{p})$.

**Theorem 3.1.1** ([BZ]). *There exists a model* $\mathrm{Sh}(K,\mathfrak{p})$ *of* $\mathrm{Sh}(K)$ *over* $\mathbb{Z}_{F,\mathfrak{p}}$ *for which there is an isomorphism of formal schemes*

$$\widehat{\mathrm{Sh}}(K,\mathfrak{p})^{\mathrm{unr}} \cong H^\times \backslash \widehat{\Omega}_\mathfrak{p}^{\mathrm{unr}} \times \widehat{B}^\times/K. \tag{3.4}$$

We now consider the special fibers at both sides of the isomorphism (3.4).

**Definition 3.1.2.** *The* weighted dual graph *associated to* $H^\times\backslash\widehat{\Omega}_\mathfrak{p}^{\mathrm{unr}} \times \widehat{B}^\times/K$ *is the graph* $G(K,\mathfrak{p}) = H^\times\backslash T_\mathfrak{p} \times \widehat{B}^\times/K$. *In other words, the vertex set of* $G(K,\mathfrak{p})$ *is given by*

$$V(G(K,\mathfrak{p})) = H^\times\backslash V(T_\mathfrak{p}) \times \widehat{B}^\times/K \tag{3.5}$$

*and its oriented edge set by*

$$OE(G(K,\mathfrak{p})) = H^\times\backslash OE(T_\mathfrak{p}) \times \widehat{B}^\times/K. \tag{3.6}$$

*The vertices and edges of this graph are weighted as follows:*

(i) *Given a vertex* $v \in V(G(K,\mathfrak{p}))$, *let* $\widetilde{v}$ *be a representative of* $v$ *in* $V(T_\mathfrak{p}) \times \widehat{B}^\times/K$. *Consider the subgroup*

$$\mathrm{PStab}(\widetilde{v}) = \mathrm{Im}(\mathrm{Stab}_{H^\times}(\widetilde{v}) \to \mathrm{P}H^\times) \tag{3.7}$$

*of* $\mathrm{P}H^\times = H^\times/F^\times$. *We define the weight of* $v$ *by* $w(v) = |\mathrm{PStab}(\widetilde{v})|$.

(ii) *Given an edge* $e \in E(G(K,\mathfrak{p}))$, *let* $\widetilde{e}$ *be an oriented edge* $\widetilde{e}$ *in* $OE(T_\mathfrak{p}) \times \widehat{B}^\times/K$ *representing* $e$. *The weight of* $e$ *is analogously defined as* $w(e) = |\mathrm{PStab}(\widetilde{e})|$.

The weights in the definition above are independent of the choice of $\widetilde{x}$ and $\widetilde{e}$.

**Definition 3.1.3.** *Let* $C$ *be a semi-stable curve over* $\mathbb{Z}_{F,\mathfrak{p}}^{\mathrm{unr}}$ *and let* $P$ *be a singular point of the special fiber of* $C$. *Then the completion of the local ring* $\mathcal{O}_{C,P}$ *is isomorphic to* $\mathbb{Z}_{F,\mathfrak{p}}^{\mathrm{unr}}[[x,y]]/(xy-\pi^w)$ *for some uniquely determined integer* $w$. *The* weight *of* $P$ *is defined to be* $w$.

**Definition 3.1.4.** *Let* $C$ *be a semi-stable curve over* $\mathbb{Z}_{F,\mathfrak{p}}^{\mathrm{unr}}$. *The* weighted dual graph *associated to* $C$ *is the dual graph of the special fiber of* $C$. *An edge* $e$ *of this graph is weighted by the weight of the ordinary double point of* $C$ *corresponding to* $e$.

Note that the vertices of the dual graph above have not been given weights.

**Definition 3.1.5.** *Let $G$ and $G'$ be weighted dual graphs. An* isomorphism *from $G$ to $G'$ is an isomorphism of graphs $G \to G'$ preserving the weights of the edges.*

**Theorem 3.1.6.** *Consider the scheme $\widehat{\mathrm{Sh}}(K, \mathfrak{p})^{\mathrm{unr}}$ from Theorem 3.1.1.*

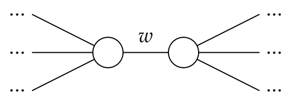
(i) *$\widehat{\mathrm{Sh}}(K, \mathfrak{p})^{\mathrm{unr}}$ is a normal scheme that is flat, proper and semistable over $\mathbb{Z}_{F,\mathfrak{p}}^{\mathrm{unr}}$.*
(ii) *The special fiber of $\widehat{\mathrm{Sh}}(K, \mathfrak{p})^{\mathrm{unr}}$ is reduced. Its components are rational curves, and all its singularities are ordinary double points. In particular, let $H$ be a connected component of the dual graph associated to $\widehat{\mathrm{Sh}}(K, \mathfrak{p})^{\mathrm{unr}}$. Then the arithmetic genus of the corresponding component of $\widehat{\mathrm{Sh}}(K, \mathfrak{p})^{\mathrm{unr}}$ equals the Betti number $1 + |E(H)| - |V(H)|$.*
(iii) *The isomorphism in Theorem 3.1.1 induces an isomorphism between $G(K, \mathfrak{p})$ and the weighted dual graph associated to $\widehat{\mathrm{Sh}}(K, \mathfrak{p})^{\mathrm{unr}}$.*

*Proof.* This follows from [Kur79, Proposition 3-2] by decomposing

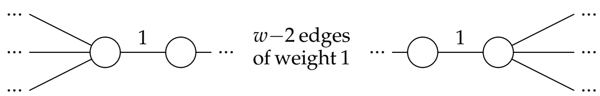
$$H^\times \backslash \widehat{\Omega}_{\mathfrak{p}}^{\mathrm{unr}} \times \widehat{B}^\times / K = \coprod_{i=1}^{h} \Gamma_i \backslash \widehat{\Omega}_{\mathfrak{p}}^{\mathrm{unr}} \tag{3.8}$$

as in Proposition 2.1.1(ii). □

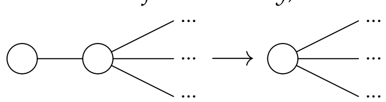
**Proposition 3.1.7** ([Kur79])**.** *Let $C$ be a curve over $\mathbb{Z}_{F,\mathfrak{p}}^{\mathrm{unr}}$ and let $\widetilde{C}$ be its minimal desingularization. Suppose that the weighted dual graph of the special fiber of $C$ contains no oriented edges equal to their own inverse. Then the dual graph of $\widetilde{C}$ can be constructed by replacing an edge of weight $w$ by a concatenation of $w$ edges of weight 1. Pictorially: a weighted edge*




*is replaced by*




*From the dual graph of $\widetilde{C}$, the dual graph of the minimal model $C_{\min}$ of $C$ can be constructed by removing vertices that belong to a unique edge, along with the edge to which they belong, until no such vertices are left. Pictorially, one inductively repeats the process*



We can therefore reconstruct the dual graph of the minimal model of $\mathrm{Sh}(K)$ over $\mathbb{Z}_{F,\mathfrak{p}}^{\mathrm{unr}}$ as soon as we can describe $G(K, \mathfrak{p})$. The following proposition generalizes part of Section 4 of [Rib90].

**Proposition 3.1.8.** *Consider the weighted dual graph $G(K, \mathfrak{p})$ constructed in Definition 3.1.2. This graph has the following properties:*

(i) *There is a bijection between $\pi_0(G(K, \mathfrak{p}))$ and the narrow class group $\mathrm{Cl}(K) = \mathrm{Cl}(N(K)\infty)$. The set $\pi_0(G(K, \mathfrak{p}))$ contains at most $|T(K\infty)|$ isomorphism classes of graphs.*
(ii) *Let $K_H$ be a compact open subgroup of $\widehat{H}^\times$ given by $K_H = K_{H,\mathfrak{p}} \times K_H^{\mathfrak{p}}$, where $K_{H,\mathfrak{p}} \subset H_{\mathfrak{p}}^\times$ is maximal and $K_H^{\mathfrak{p}} \subset \widehat{H}^{\mathfrak{p}\times}$ corresponds to $K^{\mathfrak{p}}$ under the isomorphism (3.3). Then $V(G(K, \mathfrak{p}))$ is in bijection with the disjoint union of two copies of the set $\mathrm{Pic}_r(K_H)$. An element $[\widehat{h}]$ of one of these copies is weighted by $w([\widehat{h}]) = |\mathrm{P}(\widehat{h} K_H \widehat{h}^{-1} \cap H^+)|$.*

(iii) *Let $K_H(\mathfrak{p}) = K_H(\mathfrak{p})_\mathfrak{p} \times K_H(\mathfrak{p})^\mathfrak{p}$ be a compact open subgroup of $\widehat{H}^\times$ with $K_H(\mathfrak{p})^\mathfrak{p} = K_H^\mathfrak{p}$ and $K_H(\mathfrak{p})_\mathfrak{p} = \mathcal{O}(\mathfrak{p})_\mathfrak{p}^\times$, where $\mathcal{O}(\mathfrak{p})_\mathfrak{p}$ is an arbitrary level $\mathfrak{p}$ suborder of the matrix order $\mathcal{O}(1)_\mathfrak{p}$ for which $K_{H,\mathfrak{p}} = \mathcal{O}(1)_\mathfrak{p}^\times$. Then $E(G(K,\mathfrak{p}))$ is in bijection with the set $\mathrm{Pic}_r(K_H(\mathfrak{p}))$. An element $[\widehat{h}]$ of this set is weighted by $w([\widehat{h}]) = |\mathrm{P}(\widehat{h}K_H(\mathfrak{p})\widehat{h}^{-1} \cap H^+)|$.*

(iv) *Let $w_\mathfrak{p}$ be an element of $\mathcal{O}(1)(\mathfrak{p})$ such that $\mathcal{O}(\mathfrak{p})_\mathfrak{p} = \mathcal{O}(1)_\mathfrak{p} \cap w_\mathfrak{p}\mathcal{O}(1)_\mathfrak{p} w_\mathfrak{p}^{-1}$ and let $\widehat{w}$ be an element of $\widehat{H}^\times$ whose component at $\mathfrak{p}$ is given by $w_\mathfrak{p}$ and whose components outside $\mathfrak{p}$ are trivial. Under the bijections in (ii) and (iii), the incidence relation on $G(K,\mathfrak{p})$ has the following description:*
*Let $e \in E(G(K,\mathfrak{p}))$ be given by the class $[\widehat{h}] \in \mathrm{Pic}_r(K_H(\mathfrak{p}))$. Then the vertices of $G(K,\mathfrak{p})$ connected by $e$ are given by the class $[\widehat{h}]$ in the first copy of $\mathrm{Pic}_r(K_H)$ and the class $[\widehat{h}\widehat{w}]$ in the second copy.*

(v) *$G(K,\mathfrak{p})$ does not have edges beginning and ending at the same vertex.*

(vi) *Let $v$ be a vertex of $G(K,\mathfrak{p})$ and let $E_v$ be the set of edges containing $v$. Then one has the equality*

$$\sum_{e\in E_v} \frac{w(v)}{w(e)} = \mathrm{nm}(\mathfrak{p}) + 1. \tag{3.9}$$

*Proof.* (i): Since $T_\mathfrak{p}$ is connected and $\widehat{B}^\times/K$ is totally disconnected, $\pi_0(G(K,\mathfrak{p}))$ can be described as $H^\times\backslash\widehat{B}^\times/K$. Recall that we have the following isomorphism of groups with a left $H^\times$-action:

$$\widehat{B}^\times/K \cong F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{H}^{\mathfrak{p}\times}/K_H^\mathfrak{p}. \tag{3.10}$$

The map

$$F_\mathfrak{p}^\times \times \widehat{H}^{\mathfrak{p}\times} \xrightarrow{\mathrm{id}\times\mathrm{nrd}} F_\mathfrak{p}^\times \times \widehat{F}^{\mathfrak{p}\times} \tag{3.11}$$

factorizes to give a map

$$H^\times\backslash(F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{H}^{\mathfrak{p}\times}/K_H^\mathfrak{p}) \longrightarrow F^+\backslash(F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{F}^{\mathfrak{p}\times}/\mathrm{nrd}(K_H)^\mathfrak{p}) = \mathrm{Cl}(K). \tag{3.12}$$

Since $\mathrm{nrd}(H^\times) = F^+$ by Theorem 1.2.1(i), we can apply Theorem 1.3.1 to conclude that this map is a bijection.

Indeed, let $(x_\mathfrak{p}, x^\mathfrak{p})$ represent a class in $F^+\backslash(F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{F}^{\mathfrak{p}\times}/\mathrm{nrd}(K)^\mathfrak{p})$. Because of Theorem 1.2.1(i), the fiber above $[x_\mathfrak{p}, x^\mathfrak{p}]$ is non-empty. Let $(h_\mathfrak{p}, h^\mathfrak{p})$ be a representative of an element of this fiber. Then the equality $\mathrm{nrd}(H^\times) = F^+$ from Theorem 1.2.1(ii) implies that the complete fiber above $[h_\mathfrak{p}, h^\mathfrak{p}]$ is given by the image of $\{h_\mathfrak{p}\} \times \widehat{H}^{1\mathfrak{p}}x^\mathfrak{p}$ in the double quotient $H^\times\backslash(F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{H}^{\mathfrak{p}\times}/K_H^\mathfrak{p})$.

Since the algebra $H$ splits at the finite place $\mathfrak{p}$, Corollary 1.3.2(ii) implies that $H^1\widehat{H}_\mathfrak{p}^1$ is dense in $\widehat{H}^1$. This is the same as saying that $H^1$ is dense in $\widehat{H}^{\mathfrak{p}1}$. Since $K_H^\mathfrak{p}$ is open in $\widehat{H}^\mathfrak{p}$, so is $K_H^{\mathfrak{p}1}$ in $\widehat{H}^{\mathfrak{p}1}$. Therefore the quotient $H^1\backslash\widehat{H}^{\mathfrak{p}1}h^\mathfrak{p}/K_H^{\mathfrak{p}1}$, and hence the image of $\{h_\mathfrak{p}\} \times \widehat{H}^{\mathfrak{p}1}h^\mathfrak{p}$ in $H^\times\backslash(F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{H}^{\mathfrak{p}\times}/K_H^\mathfrak{p})$, is reduced to one element. This completes the proof of the first part of (i); the second part is analogous to the proof of Proposition 2.1.1(iii).

(ii): As a set with a left $H_\mathfrak{p}^\times$-action, we have an isomorphism

$$V(T_\mathfrak{p}) \cong H_\mathfrak{p}^\times/F_\mathfrak{p}^\times K_{H,\mathfrak{p}}. \tag{3.13}$$

Therefore

$$V(G(K,\mathfrak{p})) \cong H^\times\backslash(H_\mathfrak{p}^\times/F_\mathfrak{p}^\times K_{H,\mathfrak{p}} \times F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{H}^{\mathfrak{p}\times}/K_H^\mathfrak{p}). \tag{3.14}$$

The map

$$\widehat{H}^\times/K_H = H_\mathfrak{p}^\times/K_{H,\mathfrak{p}} \times \widehat{H}^{\mathfrak{p}\times}/K_H^\mathfrak{p} \longrightarrow H_\mathfrak{p}^\times/F_\mathfrak{p}^\times K_{H,\mathfrak{p}} \times F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{H}^{\mathfrak{p}\times}/K_H^\mathfrak{p}$$
$$[h_\mathfrak{p}, h^\mathfrak{p}] \longmapsto [h_\mathfrak{p}, \mathrm{nrd}(h_\mathfrak{p}), h^\mathfrak{p}] \tag{3.15}$$

is $H^\times$-equivariant and injective. The image of (3.15) consists of the $[h_\mathfrak{p}, x_\mathfrak{p}, h^\mathfrak{p}]$ for which $v(\mathrm{nrd}(h_\mathfrak{p})) \equiv v(x_\mathfrak{p}) \pmod 2$. Let $w_\mathfrak{p}$ be as in part (iv) of the proposition. Then we can combine the map above with the similar map

$$\widehat{H}^\times/K_H = H_\mathfrak{p}^\times/K_{H,\mathfrak{p}} \times \widehat{H}^{\mathfrak{p}\times}/K_H^\mathfrak{p} \longrightarrow H_\mathfrak{p}^\times/F_\mathfrak{p}^\times K_{H,\mathfrak{p}} \times F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{H}^{\mathfrak{p}\times}/K_H^\mathfrak{p}$$
$$[h_\mathfrak{p}, h^\mathfrak{p}] \longmapsto [h_\mathfrak{p}, \mathrm{nrd}(w_\mathfrak{p})^{-1}\mathrm{nrd}(h_\mathfrak{p}), h^\mathfrak{p}]. \tag{3.16}$$

The image of (3.16) consists of the classes $[h_\mathfrak{p}, x_\mathfrak{p}, h^\mathfrak{p}]$ for which $v(\mathrm{nrd}(h_\mathfrak{p})) \equiv v(x_\mathfrak{p}) + 1 \pmod 2$. Upon modding out $H^\times$, we obtain an isomorphism

$$\coprod_{i=1}^{2} H^\times\backslash\widehat{H}^\times/K_H \xrightarrow{\sim} H^\times\backslash(H_\mathfrak{p}^\times/F_\mathfrak{p}^\times K_{H,\mathfrak{p}} \times F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{H}^{\mathfrak{p}\times}/K_H^\mathfrak{p}). \tag{3.17}$$

Keeping track of the stabilizers under (3.17) yields (ii). The choice of maximal compact subgroup $K_{H,\mathfrak{p}}$ is irrelevant, as it corresponds to a change of base point on $T_\mathfrak{p}$.

(iii): There is an isomorphism of sets with a left $H_\mathfrak{p}^\times$-action

$$E(T_\mathfrak{p}) \cong H_\mathfrak{p}^\times/N_{B_\mathfrak{p}^\times}(K_H(\mathfrak{p})_\mathfrak{p}). \tag{3.18}$$

One now essentially repeats the argument in (ii). We end up with a single copy of $H^\times\backslash\widehat{H}^\times/K_H(\mathfrak{p})_\mathfrak{p} = \mathrm{Pic}_r(K_H)$ because the reduced norm map

$$\mathrm{nrd} : N_{B_\mathfrak{p}^\times}(K_H(\mathfrak{p})_\mathfrak{p}) \longrightarrow \mathbb{Z}_{F,\mathfrak{p}}^\times \tag{3.19}$$

surjects (*cf.* Proposition 1.4.2(i)).

(iv) Under the isomorphisms (3.13) and (3.18) in (ii) and (iii), an edge of $T_\mathfrak{p}$ represented by an element $[h_\mathfrak{p}]$ of $H_\mathfrak{p}^\times/F_\mathfrak{p}^\times N_{B_\mathfrak{p}^\times}(K_H(\mathfrak{p})_\mathfrak{p})$ connects the pair of vertices of $T_\mathfrak{p}$ represented by the classes $[h_\mathfrak{p}]$ and $[h_\mathfrak{p} w_\mathfrak{p}]$ in $H_\mathfrak{p}^\times/F_\mathfrak{p}^\times K_{H,\mathfrak{p}}$. So let $[\widehat{h}] = [h_\mathfrak{p}, h^\mathfrak{p}]$ be an element of $H^\times\backslash\widehat{H}^\times/K_H(\mathfrak{p})$. This gives rise to the edge $[h_\mathfrak{p}, \mathrm{nrd}(h_\mathfrak{p}), h^\mathfrak{p}] \in E(G(K,\mathfrak{p}))$. If we choose our isomorphisms as above, this edge connects the vertices $[h_\mathfrak{p}, \mathrm{nrd}(h_\mathfrak{p}), h^\mathfrak{p}]$ and $[h_\mathfrak{p} w_\mathfrak{p}, \mathrm{nrd}(h_\mathfrak{p}), h^\mathfrak{p}]$ in $V(G(K,\mathfrak{p}))$. Under the bijection in (ii), the class $[\widehat{h}]$ in the first copy of $H^\times\backslash\widehat{H}^\times/K_H$ represents the former vertex, while the latter corresponds to the class $[\widehat{h}w(\mathfrak{p})]$ in the second.

(v): Consider a vertex of $G(K,\mathfrak{p})$ represented by $(h_\mathfrak{p}, x_\mathfrak{p}, h^\mathfrak{p})$. Its neighboring vertices are then represented by elements $(h'_\mathfrak{p}, x_\mathfrak{p}, h^\mathfrak{p})$ for which $v(\mathrm{nrd}(h'_\mathfrak{p})) \not\equiv v(\mathrm{nrd}(h_\mathfrak{p})) \pmod 2$. Therefore the two vertices are on different sides in the decomposition on the left hand side of (3.17).

(vi): Applying the decomposition (3.8), this statement follows from generalities on group actions on $T_\mathfrak{p}$. □

**Proposition 3.1.9.** *Let $\mathcal{O}$ be an order of $B$ and let $K = \widehat{\mathcal{O}}^\times$. Suppose that $\widehat{\mathcal{O}} = \mathcal{O}_\mathfrak{p} \times \widehat{\mathcal{O}}_\mathfrak{p}$, where $\widehat{\mathcal{O}}^\mathfrak{p} \subset \widehat{B}^\mathfrak{p}$ and where $\mathcal{O}_\mathfrak{p} \subset B_\mathfrak{p}$ is maximal at $\mathfrak{p}$. Let $\mathcal{O}_H$ be an order of $H$ for which $\widehat{\mathcal{O}}_H = \mathcal{O}_{H,\mathfrak{p}} \times \widehat{\mathcal{O}}_H^\mathfrak{p}$, where $\mathcal{O}_{H,\mathfrak{p}}$ is a maximal order of $H_\mathfrak{p}$ and where $\widehat{\mathcal{O}}_H^\mathfrak{p}$ corresponds to $\widehat{\mathcal{O}}^\mathfrak{p}$ under the isomorphism chosen in (3.3).*

(i) *The vertex set $V(G(K,\mathfrak{p}))$ is in bijection with two copies of $\mathrm{Pic}_r(\mathcal{O}_H)$. An element $[I]$ of one of these copies is weighted by $w([I]) = |\mathcal{O}_l(I)^\times/\mathbb{Z}_F^\times|$.*
(ii) *Let $\mathcal{O}_H(\mathfrak{p})$ be a level $\mathfrak{p}$ suborder of $\mathcal{O}_H$. The edge set $E(G(K,\mathfrak{p}))$ is in bijection with $\mathrm{Pic}_r(\mathcal{O}_H(\mathfrak{p}))$. An element $[I(\mathfrak{p})]$ of this set is weighted by $w([I(\mathfrak{p})]) = |\mathcal{O}_l(I(\mathfrak{p}))^\times/\mathbb{Z}_F^\times|$.*

(iii) *There exists a unique order $\mathcal{O}'_H$ of $H$ such that $\mathcal{O}_H(\mathfrak{p}) = \mathcal{O}_H \cap \mathcal{O}'_H$. There exists a unique lattice $I_0 \subset \mathcal{O}_H$ of level $\mathfrak{p}^2$ such that $\mathcal{O}_l(I_0) = \mathcal{O}'_H$ and $\mathcal{O}_r(I_0) = \mathcal{O}_H$. Under the bijections in (i) and (ii), the incidence relation on $G(K,\mathfrak{p})$ has the following description:*
*Let $e \in E(G(K,\mathfrak{p}))$ be represented by the ideal class $[I(\mathfrak{p})] \in \mathrm{Pic}_r(\mathcal{O}_H(\mathfrak{p}))$. Then the edge $e$ connects the vertex of $G(K,\mathfrak{p})$ given by the ideal class $[I(\mathfrak{p})\mathcal{O}_H]$ in the first copy of $\mathrm{Pic}_r(\mathcal{O}_H)$ with the ideal class $[I(\mathfrak{p})I_0]$ in the second.*

*Proof.* This follows from Proposition 3.1.8 in light of the local-global correspondence for ideals in quaternion algebras (see [Vig80, Proposition III.5.1]). □

*Remark* 3.1.10. The dual graph $G(K,\mathfrak{p})$ can be calculated if $K = \widehat{\mathcal{O}}^\times$ for an Eichler order $\mathcal{O}$ using the algorithms in [KV10]. Note that the dual graph does not depend on the choice of $\mathcal{O}_H$ and $\mathcal{O}_H(\mathfrak{p})$ if the level of $\mathcal{O}$ is squarefree, as all Eichler orders of squarefree level are locally conjugate (*cf.* Proposition 1.1.1).

Let $K = \widehat{\mathcal{O}}^\times$ be given by the adèlic units of an Eichler order $\mathcal{O}$ and let $\mathcal{O}_H$, $\mathcal{O}_H(\mathfrak{p})$ and $I_0$ be as in Proposition 3.1.9. As in the proof of Proposition 3.1.8(ii), one shows that combining the factorizations of the maps

$$\begin{aligned} \widehat{H}^\times/K_H(\mathfrak{p}) &\longrightarrow H_\mathfrak{p}^\times/F_\mathfrak{p}^\times K_H(\mathfrak{p})_\mathfrak{p} \times F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{H}^{\mathfrak{p}\times}/K_H(\mathfrak{p})^\mathfrak{p} \\ [h_\mathfrak{p}, h^\mathfrak{p}] &\longmapsto [h_\mathfrak{p}, \mathrm{nrd}(h_\mathfrak{p}), h^\mathfrak{p}] \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \widehat{H}^\times/K_H(\mathfrak{p}) &\longrightarrow H_\mathfrak{p}^\times/F_\mathfrak{p}^\times K_H(\mathfrak{p})_\mathfrak{p} \times F_\mathfrak{p}^\times/\mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{H}^{\mathfrak{p}\times}/K_H(\mathfrak{p})^\mathfrak{p} \\ [h_\mathfrak{p}, h^\mathfrak{p}] &\longmapsto [h_\mathfrak{p} w_\mathfrak{p}, \mathrm{nrd}(h_\mathfrak{p}), h^\mathfrak{p}] \end{aligned} \tag{3.21}$$

yields a bijection between $OE(G(K,\mathfrak{p}))$ and the disjoint union of two copies of $\mathrm{Pic}_r(\mathcal{O}_H(\mathfrak{p}))$. An element $[I(\mathfrak{p})]$ of the first copy of $\mathrm{Pic}_r(\mathcal{O}_H(\mathfrak{p}))$ corresponds to the oriented edge starting at the class $[I(\mathfrak{p})\mathcal{O}_H]$ in the first copy of $\mathrm{Pic}_r(\mathcal{O}_H)$ and terminating at the class $[I(\mathfrak{p})I_0]$ in the second, while the class $[I(\mathfrak{p})]$ in the second copy of $\mathrm{Pic}_r(\mathcal{O}_H(\mathfrak{p}))$ corresponds to the inverse of this oriented edge.

We now describe the action of the Atkin–Lehner automorphisms of $\mathrm{Sh}(K)$ on $OE(G(K,\mathfrak{p}))$. For this, it suffices to describe the action of the automorphisms $a(\mathfrak{q})$ for prime ideals $\mathfrak{q}$.

**Proposition 3.1.11.** *Let $K = \widehat{\mathcal{O}}^\times$, where $\mathcal{O}$ is an Eichler order of $B$ of squarefree level $\mathfrak{N}$ and let the corresponding Eichler orders $\mathcal{O}_H$ and $\mathcal{O}_H(\mathfrak{p})$ of $H$ be as in Proposition 3.1.9.*

*Let $\mathfrak{q}$ be a prime dividing $\mathfrak{p}\mathfrak{N}$. Then the action of the Atkin–Lehner automorphism $a(\mathfrak{q})$ on the oriented edge set $OE(G(K,\mathfrak{p}))$ of $G(K,\mathfrak{p})$ is as follows.*

(i) *Let $\mathfrak{q}|\mathfrak{N}$. Then there exists a unique two-sided $\mathcal{O}_H(\mathfrak{p})$-ideal $I_0(\mathfrak{q}) \subset \mathcal{O}_H(\mathfrak{p})$ of level $\mathfrak{q}^2$. The automorphism $a(\mathfrak{q})$ sends an oriented edge $[I(\mathfrak{p})]$ in a copy of $\mathrm{Pic}_r(\mathcal{O}_H(\mathfrak{p}))$ to the oriented edge $[I(\mathfrak{p})I_0(\mathfrak{q})]$ in the same copy.*

(ii) *Let $\mathfrak{q} = \mathfrak{p}$. Then there exists a unique two-sided $\mathcal{O}_H(\mathfrak{p})$-ideal $I_0(\mathfrak{p}) \subset \mathcal{O}_H(\mathfrak{p})$ of level $\mathfrak{p}^2$. The automorphism $a(\mathfrak{p})$ sends an oriented edge $[I(\mathfrak{p})]$ in a copy of $\mathrm{Pic}_r(\mathcal{O}_H(\mathfrak{p}))$ to the oriented edge $[I(\mathfrak{p})I_0(\mathfrak{p})]$ in the other copy.*

*Finally, let $A$ be a group of Atkin–Lehner automorphisms and let $\overline{G}(K,\mathfrak{p}) = G(K,\mathfrak{p})/A$ be the corresponding weighted quotient graph of $G(K,\mathfrak{p})$. Given a vertex or oriented edge $\overline{x}$ of $\overline{G}(K,\mathfrak{p})$ represented by a vertex or oriented edge $x$ of $G(K,\mathfrak{p})$, we have $w(\overline{x}) = |\mathrm{Stab}_A(x)|w(x)$.*

*Proof.* (i): On $G(K,\mathfrak{p})$, the automorphism $a(\mathfrak{q})$ induces the bijection

$$\begin{aligned} T_\mathfrak{p} \times \widehat{B}^\times/K &\longrightarrow T_\mathfrak{p} \times \widehat{B}^\times/K \\ (t, \widehat{b}) &\longmapsto (t, \widehat{b}\widehat{n}_B(\mathfrak{q})), \end{aligned} \tag{3.22}$$

where $\widehat{n}_B(\mathfrak{q})$ is as in the discussion after Proposition 2.1.1. Let $\widehat{n}(\mathfrak{q})$ be an element of $\widehat{\mathcal{O}}_H$ whose component at $\mathfrak{p}$ is trivial and whose components outside $\mathfrak{p}$ correspond to the components of $\widehat{n}_B(\mathfrak{q})$ under (3.3). Then under the isomorphism of $H^\times$-sets

$$\widehat{B}^\times / K \cong F_{\mathfrak{p}}^\times / \mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{H}^{\mathfrak{p}\times} / K_H^{\mathfrak{p}}, \tag{3.23}$$

right multiplication by $\widehat{n}_B(\mathfrak{q})$ on the left hand side corresponds to right multiplication by $\widehat{n}(\mathfrak{q})$ on the right hand side. We let $I_0(\mathfrak{q})$ be the two-sided $\widehat{\mathcal{O}}_H(\mathfrak{p})$-ideal $H \cap \widehat{n}(\mathfrak{q})\widehat{\mathcal{O}}_H(\mathfrak{p})$.

Right multiplication by $\widehat{n}(\mathfrak{q})$ does not interchange the copies of $\mathrm{Pic}_r(\mathcal{O}_H)$ constituting $OE(G(K,\mathfrak{p}))$ since the norm of $\widehat{n}(\mathfrak{q})$ is trivial at $\mathfrak{p}$. Let $[I(\mathfrak{p})]$ be an element of a copy of $\mathrm{Pic}_r(\mathcal{O}_H(\mathfrak{p}))$ that is represented by $\widehat{h} \in \widehat{H}^\times$. Then $\widehat{n}(\mathfrak{q})$ sends $[I(\mathfrak{p})]$ to the ideal class in the same copy corresponding to the adèlic element $\widehat{h}\widehat{n}(\mathfrak{q})$. But since $\widehat{n}(\mathfrak{q})$ normalizes $\widehat{\mathcal{O}}_H(\mathfrak{p})^\times$, we have

$$H \cap \widehat{h}\widehat{n}(\mathfrak{q})\widehat{\mathcal{O}}_H(\mathfrak{p}) = (H \cap \widehat{h}\widehat{\mathcal{O}}_H(\mathfrak{p}))(H \cap \widehat{n}(\mathfrak{q})\widehat{\mathcal{O}}_H(\mathfrak{p})) = I(\mathfrak{p})I_0(\mathfrak{q}) \tag{3.24}$$

by [Vig80, Proposition III.5.1].

(ii): This time we can take $\widehat{n}(\mathfrak{p})$ to be as in Proposition 3.1.8(iv). The corresponding two-sided ideal $I_0(\mathfrak{p})$ is given by $H \cap \widehat{n}(\mathfrak{p})\widehat{\mathcal{O}}_H(\mathfrak{p})$. Let $\widehat{h}$ be an element of $H$ giving rise to an ideal class $[I(\mathfrak{p})]$ in the first copy of $\mathrm{Pic}_r(\mathcal{O}_H(\mathfrak{p}))$. Under the chosen bijections, it corresponds to the element $[h_{\mathfrak{p}}, \mathrm{nrd}(h_{\mathfrak{p}}), h^{\mathfrak{p}}]$ of $H_{\mathfrak{p}}^\times / F_{\mathfrak{p}}^\times K_H(\mathfrak{p})_{\mathfrak{p}} \times F_{\mathfrak{p}}^\times / \mathbb{Z}_{F,\mathfrak{p}}^\times \times \widehat{H}^{\mathfrak{p}\times} / K_H(\mathfrak{p})^{\mathfrak{p}}$. The automorphism $a(\mathfrak{p})$ sends $[h_{\mathfrak{p}}, \mathrm{nrd}(h_{\mathfrak{p}}), h^{\mathfrak{p}}]$ to the class $[h_{\mathfrak{p}}, \mathrm{nrd}(h_{\mathfrak{p}})\mathrm{nrd}(\widehat{n}(\mathfrak{p})), h^{\mathfrak{p}}] = [(h_{\mathfrak{p}}\widehat{n}(\mathfrak{p}))\widehat{n}(\mathfrak{p}), \mathrm{nrd}(h_{\mathfrak{p}}\widehat{n}(\mathfrak{p})), h^{\mathfrak{p}}]$. This is the image of the element $\widehat{h}\widehat{n}(\mathfrak{p})$ of the second copy of $\mathrm{Pic}_r(\mathcal{O}_H(\mathfrak{p}))$. Arguing as in (i), one shows that it corresponds to the ideal class $[I(\mathfrak{p})I_0(\mathfrak{p})]$ in $\mathrm{Pic}_r(\mathcal{O}_H(\mathfrak{p}))$. The proof for the second copy of $\mathrm{Pic}_r(\mathcal{O}_H(\mathfrak{p}))$ is analogous.

The final statement of the proposition is a straightforward consequence of the definitions (*cf.* the proof of Proposition 3.1.8(vi)). □

We now give some properties of the graph $G(K', \mathfrak{p})$ for $K'$ as in (2.29):

**Proposition 3.1.12.** *Let $\mathcal{O}$ be an order of $B$ and let $K = \widehat{\mathcal{O}}^\times$. Let $N = \mathrm{nrd}(\widehat{\mathcal{O}}^\times)$, choose $N'$ as in Lemma 2.2.2, and consider $K' = \widehat{\mathcal{O}}^\times \cap \mathrm{nrd}^{-1}(N')$. Suppose that $K'$ is maximal at $\mathfrak{p}$, that is, suppose that $\mathcal{O}$ and $N'$ are maximal at $\mathfrak{p}$. Then we have the following:*

(i) *The fiber of the canonical projection map $V(G(K',\mathfrak{p})) \to V(G(K,\mathfrak{p}))$ above a vertex $v$ of $G(K,\mathfrak{p})$ corresponding to a right $\mathcal{O}_H$-ideal $I$ consists of $n/m$ vertices of weight $w(v)/m$, where $n = |N/N'| = |(\mathbb{Z}_F^+ \cap N)/\mathbb{Z}_F^{\times 2}|$ and $m = |\mathrm{nrd}(\mathcal{O}_l(I)^\times)/\mathbb{Z}_F^{\times 2}|$.*

(ii) *A similar statement holds for the edge set.*

*The weighted dual graph associated to $\mathrm{Sh}(K',\mathfrak{p})$ is independent of the choice of $N'$.*

*Proof.* We prove (i): the proof of (ii) is similar. Let $\widehat{h}K$ be a right coset representing the ideal $I$. The fiber in question is given by the image of $\widehat{h}K$ in the double quotient $\mathrm{Pic}_r(K') = H^\times \backslash \widehat{H}^\times / K'$.

The left stabilizer of the subset $\widehat{h}K$ of $\widehat{H}^\times / K'$ is given by $\mathcal{O}_l(I)^\times$. We determine the action of this stabilizer on the individual elements of the decomposition

$$\widehat{h}K = \coprod_{[k] \in K/K'} \widehat{h}K'k. \tag{3.25}$$

By Lemma 2.2.2, there are canonical isomorphisms

$$(\mathbb{Z}_F^+ \cap N)/\mathbb{Z}_F^{\times 2} = (\mathbb{Z}_F^+ \cap N)/(\mathbb{Z}_F^+ \cap N') \xrightarrow{\sim} N/N' \xleftarrow{\sim} K/K'. \tag{3.26}$$

which respect the natural left actions of the elements of $\mathcal{O}_l(I)^\times$. We can therefore conclude the argument by remarking that an element $[h]$ of $\mathcal{O}_l(I)^\times$ fixes the element $\mathrm{nrd}(\widehat{h})N'$ of the $\mathcal{O}_l(I)^\times$-module $N/N'$ if and only if $\mathrm{nrd}(h) \in \mathbb{Z}_F^{\times 2}$.

The final part of the proof is analagous to that of Proposition 2.2.3(i). □

*Remark* 3.1.13. The incidence relation on the dual graph $G(K', \mathfrak{p})$ seems to be harder to describe globally. Fortunately, in all the cases that we consider in Section 4, the graph $G(K', \mathfrak{p})$ could be reconstructed from $G(K, \mathfrak{p})$ using Proposition 3.1.12 and the demands furnished by parts (i), (iv) and (v) of Proposition 3.1.8.

Returning to general compact open $K$, let $\overline{G}(K, \mathfrak{p})$ be the weighted dual graph obtained from $G(K, \mathfrak{p})$ by the process in Proposition 3.1.7.

**Proposition 3.1.14.** *Let $\overline{H}_0$ be a connected component of $\overline{G}(K, \mathfrak{p})$. Suppose that $\overline{H}_0$ has genus* 1. *Then*

(i) *All components of* $\mathrm{Sh}(K)$ *over $F_K$ have genus* 1;
(ii) *The Jacobian $J_0(K)$ has multiplicative reduction at the primes of $F_K$ above $\mathfrak{p}$;*
(iii) *Denoting by the primes of $F_K$ over $\mathfrak{p}$ by $\mathfrak{P}$, there is a bijection of indexed sets*

$$\{v_{\mathfrak{P}}(j(J_0(K))) : \mathfrak{P}|\mathfrak{p}\} \longleftrightarrow \left\{ -\sum_{e \in E(\overline{H})} w(e) : \overline{H} \in \pi_0(\overline{G}(K, \mathfrak{p})) \right\}. \tag{3.27}$$

*Proof.* (i): Clear from the transitivity of the Galois action in Proposition 2.1.3.

(ii): This follows from Theorem 3.1.6(ii) by the Néron-Kodaira classification.

(iii): By maximality of $K$ at $\mathfrak{p}$, the extension $F_K|F$ is unramified at $\mathfrak{p}$. Using the transitivity of the Galois action, we therefore obtain (iii) from (ii) by applying Tate's algorithm to the Jacobians of the curves in the decomposition (2.8). □

*Remark* 3.1.15. The action of the Frobenius at $\mathfrak{p}$ on the right hand side of (3.27) is described in [BZ]. Under the isomorphism $\pi_0(\overline{G}(K, \mathfrak{p})) \to \mathrm{Cl}(K)$, it is the reciprocal of the action in Theorem (2.1.3).

3.2. **Searching $Y_0(p)$.** We now apply the results from the previous section to find a conjectural equation for $J_0(K)$. Our approach is much indebted to the method in [DD08], the difference being that we use the information on the valuations of $j(J_0(K))$ to narrow down the range of our search.

Abstractly, our problem is the following. Suppose that we are given the following data on an elliptic curve $E$ over a number field $L$:

**Data:**

(i) The primes $M$ of bad reduction of $E$;
(ii) A subset $S$ of the primes of multiplicative reduction of $E$, along with a list $W$ of valuations of $j(E)$ at these primes;
(iii) The traces of Frobenius $\{t_{\mathfrak{p}}(E) | \mathfrak{p} \in P\}$ of $E$ at a set $P$ of primes of $L$.

For $E = J_0(K)$, the set $M$ can be determined using Theorem 2.1.6, the sets $S$ and $W$ by using the results from the previous section, and the traces at (iii) by the methods in [Voi10].

Our strategy in the upcoming Algorithm 3.2.1 is to find an equation for $E$ by first calculating a corresponding point on a classical modular curve $Y_0(p)$ (parametrizing elliptic curves with a $p$-isogeny) for some small prime $p$. For such a point to exist, either $E$ or one of its twists should admit a $p$-isogeny over $L$.

In the cases $E = J_0(K)$, $L = F_K$ in Section 4, we can usually take take $p = 2$ due to the presence of Atkin–Lehner involutions on $E = J_0(K)$ (*cf.* Proposition 2.3.5). In the cases where these isogenies were not available, there is always a prime $p \leq 17$

such that the point counts $\mathrm{nm}(\mathfrak{p}) + 1 - t_{\mathfrak{p}}$ of some twist of $J_0(K)$ were divisible by $p$ for all small primes $\mathfrak{p}$ of $L$. This, too, suggests the presence of a $p$-isogeny.

The cases $p \in \{11, 17\}$ can be dealt with by searching the finitely generated group of $L$-points of the elliptic curve $X_0(p)$. We now describe how to deal with the cases $p \in \{2, 3, 5, 7, 13\}$ where $Y_0(p)$ has genus 0. By using part (i) and (ii) of the Data, we shall restrict the subset $Y_0(p)(L)$ in which we search for the point corresponding to $E$.

There exists a model $C \subset \mathbb{A}^2_L$ of $Y_0(p)$ of the form $C : uj = f(u)$, where $f$ is a monic integral polynomial of degree $p + 1$ whose constant term $c_0$ is a strictly positive power $p^{v_0}$ of $p$. For a fixed pair $(u, j) \in C(L)$, the value of $j$ equals the $j$-invariant of the corresponding geometric isomorphism class of elliptic curves over $\overline{L}$. An equation for a curve in this class can be found by using the "universal" elliptic curve

$$E(j) : y^2 + xy = x^3 - \frac{36}{j - 1728}x - \frac{1}{j - 1728}. \tag{3.28}$$

We parametrize $\mathbb{G}_m \to Y_0(p)$ by

$$u \longmapsto (u, j) = (u, f(u)/u). \tag{3.29}$$

Now $\mathbb{G}_m(L) = L^\times$ is not a finitely generated group. However, using part (i) of the Data, we can reduce to searching a finitely generated subgroup due to two restrictions that we shall describe presently. For the remainder of this discussion, let $(u, j)$ be the element of $Y_0(p)(L)$ corresponding to $E$.

**First restriction (at $\mathfrak{p} \notin M$).** Let $P_p$ be the set of primes of $L$ above $p$. First consider a prime $\mathfrak{p}$ that is not in $M \cup P_p$ and let $v = v_{\mathfrak{p}}(u)$ be the valuation of $u$ at $\mathfrak{p}$.

Suppose that $v > 0$. Then the terms with a factor $u$ in the numerator of (3.29) all have strictly positive valuation, whereas $v_{\mathfrak{p}}(c_0) = 0$. Hence $v_{\mathfrak{p}}(j) = -v < 0$. We get a contradiction with the hypothesis $\mathfrak{p} \notin M$.

Similarly, we see that $v < 0$ cannot occur. Hence $v = 0$. In the search for $(u, j)$, we can therefore restrict to the image under (3.29) of the finitely generated group $\mathbb{Z}_L(M \cup P_p)^\times \subset L^\times$, where $\mathbb{Z}_L(M \cup P_p)$ denotes the ring of $M \cup P_p \cup \infty$-integers of $L$.

We can generalize the considerations above. Take a prime $\mathfrak{p}$ in $P_p$ not in $M$. Then we can rule out $v < 0$ as above. On the other hand, suppose $v > v_0 e_{\mathfrak{p}}$, where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$ over $p$. Then the non-constant terms in the numerator of (3.29) all have valuation strictly larger than $v_{\mathfrak{p}}(c_0) = v_0 e_{\mathfrak{p}}$. Therefore $v_{\mathfrak{p}}(j) = v_{\mathfrak{p}}(c_0) - v_{\mathfrak{p}}(u) = v_0 e_{\mathfrak{p}} - v < 0$, a contradiction once again. We conclude that $0 \leq v \leq v_0 e_{\mathfrak{p}}$.

**Second restriction (at $\mathfrak{p} \in S$).** We can further restrict the search using part (ii) of the Data. Let $\mathfrak{p} \in S$ be given, and denote the given valuation at that prime by $W(\mathfrak{p})$. Again let $v = v_{\mathfrak{p}}(u)$.

Suppose that $\mathfrak{p} \in P_p$. Then we cannot have $0 \leq v \leq v_0 e_{\mathfrak{p}}$. Indeed, then the constant term $c_0$ of $f(u)$ would have valuation $v_{\mathfrak{p}}(c_0) = v_0 e_{\mathfrak{p}} \geq v$. Since the non-constant terms also have valuation at least $v$, we would get $v_{\mathfrak{p}}(j) = v_{\mathfrak{p}}(f(u)) - v_{\mathfrak{p}}(u) \geq v - v = 0$, contradicting the multiplicative reduction of $E$. If $v > v_0 e_{\mathfrak{p}}$, then $W(\mathfrak{p}) = v(j) = v_0 e_{\mathfrak{p}} - v$ as in the proof of the first restriction, and if $v < 0$, then one calculates $W(\mathfrak{p}) = pv$.

We conclude that at the primes $\mathfrak{p} \in S$, we have

$$v = v_{\mathfrak{p}}(u) \in \{v_0 e_{\mathfrak{p}} - W(\mathfrak{p}), W(\mathfrak{p})/p\} \cap \mathbb{Z} \tag{3.30}$$

if $\mathfrak{p}$ is over $p$, and

$$v = v_{\mathfrak{p}}(u) \in \{-W(\mathfrak{p}), W(\mathfrak{p})/p\} \cap \mathbb{Z} \tag{3.31}$$

otherwise. This motivates the following algorithm.

**Algorithm 3.2.1.** *Let $E$ be an elliptic curve over a number field $L$ for which the Data at the beginning of the section are available. Let $p \in \{2, 3, 5, 7, 13\}$. Choose a parametrization of $Y_0(p)$ as in (3.29) and let $v_0$, $e_{\mathfrak{p}}$, $W(\mathfrak{p})$ and $P_p$ be as in the discussion above. Let $\mathfrak{p}_1 \dots \mathfrak{p}_m$ be the primes in $P_p - M$ and let $\mathfrak{q}_1 \dots \mathfrak{q}_n$ be the primes in $M - S$.*

*If $E$ or one of its twists admits a $p$-isogeny over $L$, then the following algorithm determines a non-empty list $L$ of conjectural equations for $E$ agreeing with the Data.*

***1.*** *Let*

$$V = \prod_{\mathfrak{p} \in S \cap P_p} \{v_0 e_{\mathfrak{p}} - W(\mathfrak{p}), W(\mathfrak{p})/p\} \cap \mathbb{Z} \times \prod_{\mathfrak{p} \in S \setminus P_p} \{-W(\mathfrak{p}), W(\mathfrak{p})/p\} \cap \mathbb{Z}. \tag{3.32}$$

*For $v \in V$, let $\mathfrak{a}_v$ be the ideal of $\mathbb{Z}_L$ whose valuations outside $S$ are trivial and whose valuations at the primes in $S$ agree with $v$. Construct the set $A = \{\mathfrak{a}_v : v \in V\}$.*

***2.*** *Compute a set of generators $u_1, \dots, u_r$ for $\mathbb{Z}_L^\times$.*

***3.*** *Choose a large integer $N$. Initialize $L = \emptyset$ and let*

$$R = \left\{ \mathfrak{a}_v \mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_m^{k_m} \mathfrak{q}_1^{l_1} \cdots \mathfrak{q}_n^{l_n} \mid \mathfrak{a}_v \in A,\ 0 \leq k_i \leq v_0 e_{\mathfrak{p}_i},\ -N \leq l_j \leq N \right\}. \tag{3.33}$$

*Choose an ideal $\mathfrak{a}$ in $R$.*

***4.*** *Test if $\mathfrak{a}$ is principal. If it is, then choose $a \in \mathbb{Z}_L$ such that $\mathfrak{a} = (a)$ and go to step 5; otherwise, go to step 9.*

***5.*** *Let $S = \{a u_1^{s_1} \cdots u_r^{s_r} | -N \leq s_t \leq N\}$. Choose an element $u$ of $S$.*

***6.*** *Calculate $j = f(u)/u$ and determine the elliptic curve $E(j)$ from (3.28).*

***7.*** *Let $M(j)$ be the set of primes of bad reduction of $E(j)$. Determine a set of representatives for the finite quotient $\mathbb{Z}_{L,M\cup M(j)}^\times / \mathbb{Z}_{L,M\cup M(j)}^{\times e}$. Here*

$$e = \begin{cases} 6 & \text{if } j = 0 \\ 4 & \text{if } j = 1728 \\ 2 & \text{otherwise.} \end{cases} \tag{3.34}$$

*Construct the set of twists $T$ of $E$ by these representatives.*

***8.*** *Add to $L$ the equations of those elements of $T$ not yet in $L$ that agree with the rest of the Data. Choose a new element $u \in S$ and return to step 6. Repeat until all elements of $S$ have been used.*

***9.*** *Choose a new ideal $\mathfrak{a}$ in $R$ and return to step 4. Repeat until all ideals in $R$ have been used.*

***10.*** *If $L$ is empty, then enlarge $N$ and return to step 3. Otherwise, return $L$.*

## 4. Conjectural models

Let $\Gamma$ be an arithmetic $(1;e)$-group, let $J^\pm(\Gamma) = \mathrm{Jac}(X^\pm(\Gamma))$ and let $E^\pm(\Gamma)$ be a canonical model of $J^\pm(\Gamma)$ as in Definition 2.3.3. We now seek to determine conjectural equations for $E^\pm(\Gamma)$ by using the following information on these curves:

- The canonical field of definition of $E^\pm(\Gamma)$ (see Remark 1.2.2),
- The field generated by $j(E^\pm(\Gamma))$ (from Theorem 2.1.7),
- The reduction properties of $E^\pm(\Gamma)$ (Theorem 2.1.6),
- The traces of Frobenius of $E^\pm(\Gamma)$ (by the methods of [Voi10]),
- The valuations of $j(E^\pm(\Gamma))$ from Proposition 3.1.14, and, if applicable,
- The values of $j(E^\pm(\Gamma))$ obtained in [Sij11].

We refer to the next section for a proof of correctness of most of these conjectural equations.

Our results are given in Table 3; this section highlights a selected few calculations of which the remaining cases are analogues. We also give minimal Weierstrass models for $E^{\pm}(\Gamma)$ for a few cases in which this model is not too involved to write down. To give these equations, we use the algebraic integers $\alpha$ from Table 1. We use the labels e$n_e$ d$n_d$ D$n_D$ r defined in the appendix. For the sake of completeness, we have included the cases with ground field $\mathbb{Q}$ from [Elk98] and [GR06].

*Remark* 4.1. Note that whenever the class group $\mathrm{Cl}(K)$ is trivial, we have $\mathrm{Sh}_0^+(K) \cong \mathrm{Sh}_0^-(K)$, whence also $X^+(K \cap B^\times) \cong X^-(K \cap B^\times)$. In the cases where the curves $X^{\pm}(\Gamma)$ are Atkin–Lehner quotients of such curves, we therefore restrict our considerations to $X(\Gamma) = X^+(\Gamma)$.

**e2d1D6i:** Let $\mathcal{O}(5) = \mathbb{Z}[\Gamma^{(2)}]$. Then $\Gamma^{(2)} = \mathcal{O}(5)^1$, and $\mathcal{O}(5)$ is of index 5 in a maximal order containing it. Hence $\mathcal{O}(5)$ is a level 5 Eichler order by Proposition 1.1.1. As such, the (1;2)-curve $X(\Gamma^{(2)}) = X(\mathcal{O}(5)^1)$ has a canonical model given by the Shimura curve $\mathrm{Sh}_0(\mathcal{O}(5))$, which is canonically defined over $\mathbb{Q}_\infty = \mathbb{Q}$ by Theorem 1.2.1. Since all Atkin–Lehner automorphisms of $\mathrm{Sh}_0(\mathcal{O}(5))$ are also defined over $\mathbb{Q}$ by Theorem 2.1.2(iii), the Jacobian $J_0(\mathcal{O}(5))$ is a canonical model of $J(\Gamma)$ by Lemmata 2.3.2 and 2.3.5. The elliptic curve $J_0(\mathcal{O}(5))$ was determined in [Elk98] and [GR06]. It is given by

$$y^2 + xy + y = x^3 - 334x - 2368. \tag{4.1}$$

**e2d1D6ii:** We know $j(J(\Gamma)) = 2^4 13^3/3^2$ from [Sij11], where it was also mentioned that $\Gamma^{(2)}$ generates a level $2^3$ non-Eichler order $\mathcal{O} = \mathbb{Z}[\Gamma^{(2)}]$ with $\Gamma^{(2)} = \mathcal{O}^1$. As in the previous case, $J_0(\mathcal{O})$ is a canonical model of $J(\Gamma)$.

There is a unique maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$ since the prime 2 divides the discriminant of the quaternion algebra $B$ associated to $\Gamma$. We have $\mathcal{O}(1)/\mathcal{O} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. As in Remark 1.2.2, one calculates that

$$\mathrm{nrd}(\widehat{\mathcal{O}}^\times) = U_2^{(2)} \times \prod_{p \neq 2} U_p^{(0)}. \tag{4.2}$$

Therefore the canonical field of definition of $\mathrm{Sh}_0(\mathcal{O})$ is the ray class extension $\mathbb{Q}_{4\infty} = \mathbb{Q}(i)$ of $\mathbb{Q}$. However, if we let $\mathcal{O}' = \mathcal{O} + 2\mathcal{O}(1)$, then $\mathcal{O}'$ satisfies $\mathfrak{p}_2\mathcal{O}(1) \subset \mathcal{O}$, and clearly $\mathcal{O}(1)/\mathcal{O}' \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Algorithm 2.3 from [Sij11] shows that $[\mathcal{O}(1)^1 : \mathcal{O}'^1] = 6 = [\mathcal{O}(1)^1 : \mathcal{O}^1]$, so in fact $\mathcal{O}'^1 = \mathcal{O}^1$. Using Corollary 1.3.5, we see that the inclusion $\mathcal{O}' \subset \mathcal{O}(1)$ corresponds to case (iv) of Proposition 1.1.4.

The elements of $\Gamma$ normalize the order $\mathcal{O} = \mathbb{Z}[\Gamma^{(2)}]$. Lemma 2.3.5 shows that the corresponding automorphisms of $\mathrm{Sh}_0(\mathcal{O})$ are induced by elements of $\widehat{B}^\times$. These elements will normalize the two-sided ideal $2\mathcal{O}(1)_2$ of the unique local maximal order $\mathcal{O}(1)_2$ at 2, hence $\mathcal{O}'$ as well by the local-global correspondence for quaternion orders. We conclude that the elements of $\Gamma$ normalize the order $\mathcal{O}'$ as well as $\mathcal{O}$. Consequently, by Lemmata 2.3.2 and 2.3.5, $J_0(\mathcal{O}')$ is also a canonical model of $J(\Gamma)$. Since $U_2^{(1)} U_2^2 = \mathbb{Z}_2^\times$, Proposition 1.1.4 gives that $\mathrm{nrd}(\widehat{\mathcal{O}}'^\times) = \widehat{\mathbb{Z}}^\times$. Hence $J_0(\mathcal{O}') = J(\mathcal{O}')$ is defined over $\mathbb{Q}_\infty = \mathbb{Q}$.

By Theorem 2.1.6, $J_0(\mathcal{O}')$ is one of 8 elliptic curves over $\mathbb{Q}$ with $j$-invariant $2^4 13^3/3^2$ having good reduction outside the primes 2 and 3. Inspection of the traces of Frobenius (as in [Voi10]) yields the correct twist, which is the strong Weil curve of conductor 24.

**e2d5D4i:** We know $j(J(\Gamma))$ from [Sij11]. Let $\mathcal{O}$ be the level $\mathfrak{p}_2^2$ non-Eichler order $\mathbb{Z}_F[\Gamma^{(2)}]$. Then as in the case e2d1D6i, $J_0(\mathcal{O})$ is a canonical model for $J(\Gamma)$. The maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$ satisfies $\mathfrak{p}\mathcal{O}(1) \subset \mathcal{O}$ and $\mathcal{O}(1)/\mathcal{O} \cong (\mathbb{Z}_F/\mathfrak{p}_2)^2$. Using Table 2 and Corollary 1.3.5, we see that we are in case (iv) of Proposition 1.1.4. As in the case e2d1D6ii, one concludes that $J_0(\mathcal{O})$ is defined over $F_\infty = F$. This time, determining the correct twist yields the model

$$y^2 = x^3 + (\alpha - 1)x^2 + (-6\alpha - 5)x + (-11\alpha - 7). \tag{4.3}$$

*Remark* 4.2. The curve (4.3) is isomorphic to its $\mathrm{Gal}(F|\mathbb{Q})$-conjugate over $F$. However, $J_0(\mathcal{O})$ does not descend to $\mathbb{Q}$. Indeed, the trace of Frobenius of $J_0(\mathcal{O})$ at the inert prime $\mathfrak{p}_3$ equals 2, which is not of the form $n^2 - 2 \cdot 3$ with $n \in \mathbb{Z}$. This technique was frequently used to determine the minimal field of definition of $E(\Gamma)$ in Table 3.

**e2d5D4ii:** In this case, $\Gamma^{(2)}$ generates a maximal order $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma^{(2)}]$ such that $\mathcal{O}(1)^1$ has signature $(0, 2, 5, 5)$. As such, it is therefore more involved to obtain a canonical model of $X(\Gamma)$ in this case: no order $\mathcal{O}$ suggests itself for which $X(\Gamma)$ can be realized as an Atkin–Lehner quotient of $Y_0(\mathcal{O})$.

We know $j(J(\Gamma)) = 5^1 211^3/2^{15}$ from [Sij11]. There are two elliptic curves over $F$ of $\mathfrak{p}_2\mathfrak{p}_5^2$ with this $j$-invariant. Corresponding equations are

$$y^2 + xy + y = x^3 + x^2 + 22x - 9 \text{ and} \tag{4.4}$$

$$y^2 + \alpha xy + (\alpha + 1)y = x^3 + (\alpha - 1)x^2 + (-111\alpha + 220)x + (-287\alpha + 528). \tag{4.5}$$

Considering Theorem 2.1.6, this suggests that we can obtain $X(\Gamma)$ using a compact open group $K \subset \widehat{B}^\times$ that is non-maximal at $\mathfrak{p}_5$ only. Let $\mathcal{O}(1)$ be the maximal order generated by $\Gamma$ and consider $C = (\widehat{\mathbb{Z}}_F + \mathfrak{p}_5\widehat{\mathcal{O}}(1))^\times$. This group equals the core of $\widehat{\mathcal{O}}(\mathfrak{p}_5)^\times$ in $\widehat{B}^\times$ for any level $\mathfrak{p}_5$ Eichler order $\mathcal{O}(\mathfrak{p}_5)$ contained in $\mathcal{O}(1)$. We have

$$\mathrm{nrd}(C) = U_{\mathfrak{p}_5}^2 \times \prod_{\mathfrak{p} \nmid 5} U_{\mathfrak{p}}^{(0)}. \tag{4.6}$$

The monodromy group $M$ of the cover $Y_0(\mathcal{O}(\mathfrak{p}_5)) \to Y_0(\mathcal{O}(1))$ is isomorphic to $\mathrm{PSL}(2, \mathbb{F}_5)$ (*cf.* Remark 2.4 in [Sij11]). Moreover, $M$ is isomorphic to the Galois group of the cover

$$Y_0(C) \longrightarrow Y_0(\mathcal{O}(1)). \tag{4.7}$$

Up to conjugation, $M$ has a unique subgroup $H$ of index 5. Using [Sij11, Algorithm 2.3], we that the corresponding cover of $Y_0(\mathcal{O}(1)) \cong X(\mathcal{O}(1)^1)$ has ramification type $((2, 2, 1), (5), (5))$. By [Sij11, Algorithm 2.5], there is a unique Belyĭ map with the ramification type above. Consequently $H$ corresponds to the cover $X(\Gamma) \to X(\mathcal{O}(1)^1)$ that was calculated in [Sij11]. We conclude that the cover $X(\Gamma) \to X(\mathcal{O}(1)^1)$ is the unique degree 5 factorization of (4.7).

A calculation in the group $\mathcal{O}(1)_{\mathfrak{p}_5}^\times / C_{\mathfrak{p}_5} \cong \mathrm{PGL}(2, \mathbb{F}_5)$ shows that up to conjugation, there is a unique index 5 subgroup $K$ of $\widehat{\mathcal{O}}(1)^\times$ containing $C$. Corollary 1.3.3 gives that the degree of the resulting cover $Y_0(K) \to Y_0(\mathcal{O}(1))$ also equals 5. Hence $X(\Gamma) \cong Y_0(K)$, realizing the arithmetic $(1;e)$-curve $X(\Gamma)$ as a Shimura curve. Since $K$ contains an element whose norm in $\mathbb{Z}_F/\mathfrak{p}_5$ is not a square, we have $\mathrm{nrd}(K) = \widehat{\mathbb{Z}}_F^\times$ in light of (4.6).

By the preceding discussion, $X(\Gamma)$ has a canonical model $\mathrm{Sh}_0(K) = \mathrm{Sh}(K)$ over $F_K = F_\infty = F$. A model for $J_0(K)$ can be determined as in the case e2d1D6ii. It is given by (4.5).

*Remark* 4.3. Although $K$ is not of the form $\mathcal{O}^\times$, as in [Voi10], it is still possible to calculate the traces of Frobenius of $J_0(K)$ by using ad hoc methods. We refer to [Sij10b] for an implementation.

**e2d8D2:** We know $j(J(\Gamma)) = 1728$ from [Sij11]. Let $\mathcal{O} = \mathbb{Z}_F[\langle \Gamma^{(2)}, \alpha\beta \rangle]$. Then $\mathcal{O}^1 = \langle \Gamma^{(2)}, \alpha\beta \rangle$. There is a unique maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$. As a $\mathbb{Z}_F$-module, $\mathcal{O}(1)/\mathcal{O} \cong (\mathbb{Z}_F/\mathfrak{p}_2^2)^2$. Using [Sij11, Algorithm 2.3], one checks that there are no orders $\mathcal{O}'$ inbetween $\mathcal{O}$ and $\mathcal{O}(1)$ with $\mathcal{O}'^1 = \mathcal{O}^1$. Nor are there such orders with $\mathcal{O}^1 = \Gamma$, since in this case $\beta$ does not come from an element of the quaternion algebra associated to $\Gamma$. We therefore use an Atkin–Lehner quotient of $\mathrm{Sh}(\mathcal{O})$ as a canonical model for $X(\Gamma)$ (*cf.* Lemma 2.3.5).

As in Remark 1.2.2, one calculates that

$$\mathrm{nrd}(\widehat{\mathcal{O}}^\times) = U_{\mathfrak{p}_2}^{(2)} \times \prod_{\mathfrak{p} \nmid \mathfrak{p}_2} U_{\mathfrak{p}}^{(0)} \tag{4.8}$$

Hence the canonical field of definition of $\mathrm{Sh}_0(\mathcal{O})$ is $F_{\mathfrak{p}_2^2\infty} = F(i)$. Using the techniques from the case e2d1D6ii, we see that $J_0(\mathcal{O})$ is the base extension to $F_\infty$ of either of the elliptic curves

$$y^2 = x^3 - x \tag{4.9}$$

$$y^2 = x^3 + x. \tag{4.10}$$

*Remark* 4.4. Over $F$, the curves (4.9) and (4.10) are isogenous but not isomorphic. These curves correspond to the $F$-factors of the Jacobian $J(\mathcal{O})$ constructed in [Hid81, Theorem 4.4]. A similar phenomenon will occur on the future occasions e2d12D3, e2d33D12, e2d229D8, e3d12D3, e3d28D18 and e6d12D66.

**e2d8D7i/ii:** These two cases are conjugate (*cf.* Theorem 2.1.7); we treat the first. If we let $\mathcal{O} = \mathbb{Z}_F[\Gamma^{(2)}]$, then $\mathcal{O}^1 = \langle \Gamma^{(2)}, \beta \rangle$. There is a maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$ such that $\mathcal{O}(1)/\mathcal{O} \cong (\mathbb{Z}_F/\mathfrak{p}_2)^2$. Using Table 2 and Corollary 1.3.5, we see that we are in case (iii) of Proposition 1.1.2. Consequently $\mathrm{nrd}(\widehat{\mathcal{O}}^\times) = \widehat{\mathbb{Z}}_F^\times$, hence the canonical field of definition of $\mathrm{Sh}_0(\mathcal{O}) = \mathrm{Sh}(\mathcal{O})$ is given by $F_\infty = F$. By Lemma 2.3.5, there is a canonical model of $J(\Gamma)$ that is 2-isogenous to $J_0(\mathcal{O})$ over $F$. So we search $X_0(2)$ in Algorithm 3.2.1. This yields the candidate

$$y^2 = x^3 + \alpha x^2 + (-2\alpha - 2)x + (-2\alpha - 3) \tag{4.11}$$

for the isogeny class of the canonical models of $J_0(\mathcal{O})$ and $J(\Gamma)$. With our methods, we cannot hope to calculate more than this isogeny class.

**e2d12D2:** In this case and the next, we have $|\mathrm{Cl}(\infty)| \neq 1$. We know $j(J(\Gamma)) = 0$ from [Sij11]. Let $\mathcal{O}(\mathfrak{p}_2) = \mathbb{Z}_F[\Gamma]$. As in the case e2d5D4i, we get that $\mathrm{nrd}(\widehat{\mathcal{O}}(\mathfrak{p}_2)^\times) = \widehat{\mathbb{Z}}_F^\times$. Using Proposition 2.2.1, we obtain $[P\mathcal{O}(\mathfrak{p}_2)^+ : P\mathcal{O}(\mathfrak{p}_2)^1] = 2$. As a result, the group $\mathcal{O}(\mathfrak{p}_2)^+$ has signature $(0;2,2,2,4)$. We therefore have to choose groups $N'$ and $K'$ for $K = \widehat{\mathcal{O}}(\mathfrak{p}_2)^\times$ as in (2.29) in order to obtain a canonical $K'$-model $\mathrm{Sh}_0(K')$ of $X(\Gamma)$ over $F_\infty = F(i)$.

We first choose

$$N' = U_{\mathfrak{p}_2}^{(2)} \times \prod_{\mathfrak{p} \nmid 2} U_{\mathfrak{p}}^{(0)}. \tag{4.12}$$

To see that $N'$ satisfies the conditions from Lemma 2.2.2, note that the representative $\alpha + 2$ of the unique non-trivial coset in $\mathbb{Z}_F^+/\mathbb{Z}_F^{\times 2}$ is not congruent to 1 modulo $\mathfrak{p}_2^2$ and that $\mathrm{Cl}(\mathfrak{p}_2^2\infty) \cong \mathrm{Cl}(\infty)$. We have $F_{K'} = F_K = F_\infty \cong F(i)$. Over this field, $\mathrm{Sh}(K')$ has $|\mathrm{Cl}(K)| = 2$ connected components. Since the elements $\{\pm 1, 1\}$ of $\{\pm 1\} \times \widehat{B}^\times$ represent different classes in $\mathrm{Cl}(K')$, these components are given by the curves $\mathrm{Sh}_0^+(K')$ and $\mathrm{Sh}_0^-(K')$. By construction, the curves $\mathrm{Sh}_0^\pm(K')$ are canonical models of $X^\pm(\Gamma)$.

We have $|T(K'\infty)| = 1$ by Proposition 1.4.1 and Proposition 2.1.4. Indeed, since $\mathcal{O}(\mathfrak{p}_2)$ is the unique level $\mathfrak{p}_2$ suborder of $\mathcal{O}(1)$, we have

$$N(K') \supset N(\widehat{\mathcal{O}}(\mathfrak{p}_2)^\times) = N(\widehat{\mathcal{O}}(\mathfrak{p}_2)) = N(\widehat{\mathcal{O}}(1)), \tag{4.13}$$

which implies our claim by Proposition 1.4.2(iii). As a result, the canonical models $\mathrm{Sh}_0^\pm(K')$ are in fact isomorphic over $F_{K'}$.

By Theorem 2.1.6, $\mathrm{Sh}(K')$ has good reduction away from $\mathfrak{p}_2$. Hence $\mathrm{Sh}_0(K')$ has good reduction outside $\mathfrak{p}_2\mathbb{Z}_{F_\infty}$. One can now determine the canonical model of $J_0(K')$ as in the case e2d1D6ii. It is the base extension to $F_\infty$ of the curve

$$y^2 = x^3 + \alpha x^2 + x + (3\alpha - 5). \tag{4.14}$$

We can also take $N'$ to equal

$$N' = U_{\mathfrak{p}_3}^2 \times \prod_{\mathfrak{p}\nmid 3} U_{\mathfrak{p}}^{(0)} \tag{4.15}$$

of $\mathbb{Z}_F$ for the odd place $\mathfrak{p}_3$. The corresponding canonical model $J_0(K')$ is the base extension to $F_\infty$ of the curve $y^2 = x^3 - 1$.

The results above show that the $K'$-model $J_0(K')$ of $J(\Gamma)$ may depend on the choice of $K'$ (*cf.* Remark 2.2.4). In some sense one can get around this non-uniqueness. The orders $\mathcal{O} = \mathbb{Z}_F[G]$ in Table 2 other than $\mathbb{Z}_F[\Gamma]$ all satisfy

$$\mathrm{nrd}(\mathcal{O}^\times) = U_{\mathfrak{p}_2}^2 \times \prod_{\mathfrak{p}\nmid 2} U_{\mathfrak{p}}^{(0)}. \tag{4.16}$$

Hence $\mathrm{P}\mathcal{O}^1 = \mathrm{P}\mathcal{O}^+$ for these orders, and we get $J_0(\mathcal{O})$ as a canonical model of $J(\Gamma)$. These choices all result in the model (4.14). This $K'$-model also has good reduction outside of $\mathfrak{p}_2$, which is as optimal as can be reasonably wished considering that we started with a quaternion algebra $B$ ramified at $\mathfrak{p}_2$ (*cf.* Theorem 2.1.6). Therefore we have used (4.14) in the final Table 3.

**e2d12D3:** We know $j(J(\Gamma)) = 2^2 193^3/3$ from [Sij11]. We saw there that the group $\Gamma^{(2)}$ generates a level $\mathfrak{p}_2^4$ non-Eichler order $\mathcal{O} = \mathbb{Z}_F[\Gamma^{(2)}]$ with $\mathcal{O}^1 = \Gamma^{(2)}$. There is a maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$ such that $\mathcal{O}(1)/\mathcal{O} \cong (\mathbb{Z}_F/\mathfrak{p}_2^2)^2$. As in Remark 1.2.2, one calculates that

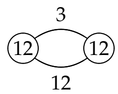
$$\mathrm{nrd}(\widehat{\mathcal{O}}^\times) = U_{\mathfrak{p}_2}^{(2)} \times \prod_{\mathfrak{p}\nmid 2} U_{\mathfrak{p}}^{(0)}, \tag{4.17}$$

which implies that $\mathrm{P}\mathcal{O}^+ = \mathrm{P}\mathcal{O}^1$ by Proposition 2.2.1. The curve $\mathrm{Sh}_0(\mathcal{O})$ is defined over $F_{\mathfrak{p}_2^2\infty} = F_\infty = F(i)$, and by Lemmata 2.3.2 and 2.3.5, $J_0(\mathcal{O})$ furnishes a canonical model for $J(\Gamma)$.

As in the case e2d1D6ii, one argues that $J_0(\mathcal{O})$ is described by the equation $y^2 = x^3 - x^2 - 64x + 220$. And as in the previous case, there is an isomorphism $J_0^+(\mathcal{O}) \cong J_0^-(\mathcal{O})$.

**e2d13D4:** The group $\Gamma$ generates a maximal order $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma]$ for which $\Gamma = \mathcal{O}(1)^1$. The canonical model $\mathrm{Sh}_0(\mathcal{O}(1)) = \mathrm{Sh}(\mathcal{O}(1))$ of $X(\Gamma)$ is defined over $F_\infty = F$. Since the finite part $\mathfrak{D}(B)^f$ of the discriminant $B$ is $G_{\mathbb{Q}}$-invariant, Corollary 2.1.8 shows that $j(J_0(\mathcal{O}(1))) \in \mathbb{Q}$. The extension $F|\mathbb{Q}$ is ramified above 13 only. Consequently, $J_0(\mathcal{O})$ is an $F$-twist of an elliptic curve over $\mathbb{Q}$ whose conductor is of the form $2^1 13^i$.

Using [Cre06], it turns out that there are exactly two twists whose traces of Frobenius agree with those obtained using the methods from [Voi10]. These twists are isogenous. Proposition 3.1.9 gives the following dual graph for $\mathrm{Sh}(\mathcal{O}(1))$ at $\mathfrak{p}_2$:

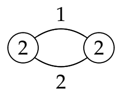



Therefore the valuation of $j(J_0(\mathcal{O}(1)))$ at $\mathfrak{p}_2$ equals $-15$ by Proposition 3.1.14. This determines which of the two aforementioned candidates is correct. A minimal Weierstrass equation for this model is given by

(4.18)
$$y^2 + \alpha xy + (\alpha+1)y = x^3 + (-\alpha-1)x^2 + (-75\alpha - 100)x + (-433\alpha - 566).$$

**e2d17D2i/ii:** These two cases are conjugate (*cf*. Theorem 2.1.7). We consider the first. The level $\mathfrak{p}_2'^2$ non-Eichler order $\mathcal{O} = \mathbb{Z}_F[\langle \Gamma^{(2)}, \alpha\beta \rangle]$ has norm 1 group $\mathcal{O}^1 = \langle \Gamma^{(2)}, \alpha\beta \rangle$. The index from Table 2, along with Corollary 1.3.5, shows that we are in case (iv) of Proposition 1.1.2. Let $\mathcal{O}(\mathfrak{p}_2')$ be the unique level $\mathfrak{p}_2'$ Eichler order inbetween $\mathcal{O}$ and a maximal order $\mathcal{O}(1)$. This order exists by Proposition 1.1.2, which also shows that we have $\mathcal{O}(\mathfrak{p}_2')^1 = \mathcal{O}^1$.

Proposition 3.1.9 shows that the dual graph of $\mathrm{Sh}(\mathcal{O}(\mathfrak{p}_2'))$ at $\mathfrak{p}_2$ is given by




Arguing as in the case e2d8D7, one shows that $J_0(\mathcal{O}(\mathfrak{p}_2'))$ admits a 2-isogeny. A search of $X_0(2)$ using Algorithm 3.2.1 returns a conjectural model $C$ for $J_0(\mathcal{O}(\mathfrak{p}_2'))$. Taking isogenies of prime degree $\leq 50$, we found two curves in the $F$-isogeny class of $C$ whose $j$-invariants have valuation $-3$ at $\mathfrak{p}_2$. In the next section, we prove that these are indeed the only two such curves. Corresponding equations are given by

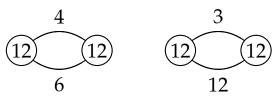
$$y^2 + xy + \alpha y = x^3 + (-\alpha+1)x^2 + (606\alpha - 1553)x + (12977\alpha - 33243), \tag{4.19}$$

$$y^2 + xy + (\alpha+1)y = x^3 + \alpha x^2 + (981\alpha - 2517)x + (23628\alpha - 60528). \tag{4.20}$$

The matrices $\alpha$ and $\beta$ normalize the order $\mathcal{O}(\mathfrak{p}_2')$ as well as $\mathcal{O}$. Therefore, using Lemma 2.3.5, we see that a canonical model of $J(\Gamma)$ can be recovered from the correct model above by taking a 2-isogeny over $F$. Fortunately, (4.19) and (4.20) both have a unique such isogeny, and we end up with the same quotient either way. A minimal Weierstrass equation is given by

$$y^2 + xy + (\alpha+1)y = x^3 + \alpha x^2 + (61\alpha - 157)x + (348\alpha - 896). \tag{4.21}$$

**e2d21D4:** In this case $|\mathrm{Cl}(\infty)| \neq 1$. The maximal order $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma^{(2)}]$ has $\mathcal{O}(1)^1 = \langle \Gamma^{(2)}, B \rangle$. The curve $\mathrm{Sh}_0(\mathcal{O}(1))$ is canonically defined over $F_\infty = F(w)$, where $w = w_{-7}$. By Proposition 3.1.9, the dual graph of $\mathrm{Sh}(\mathcal{O}(1))$ at $\mathfrak{p}_2$ is the following:




By Proposition 3.1.6(ii), $\mathrm{Sh}_0(\mathcal{O}(1))$ has genus 1, hence $\mathcal{O}(1)^+$ has signature $(1;2)$. Consequently $P\mathcal{O}(1)^+ = \Gamma$. The connected components of $\mathrm{Sh}(\mathcal{O}(1))$ are given by $\mathrm{Sh}_0^+(\mathcal{O}(1))$ and $\mathrm{Sh}_0^-(\mathcal{O}(1))$, as in the case e2d12D2. We can use the curves $\mathrm{Sh}_0^\pm(\mathcal{O}(1))$ as canonical models of $X^\pm(\Gamma)$.

Let $K'$ be as in (2.29). Then the inclusion $K' \subset \widehat{\mathcal{O}}(1)^\times$ gives rise to a 2-isogeny $J_0(K') \to J_0(\mathcal{O}(1))$ over $F_\infty$. There turns out to be a point in $X_0(2)(\mathbb{Q}(w)) \subset$

$X_0(2)(F_\infty)$ that gives rise to the correct traces of Frobenius up to a minus sign. A corresponding elliptic curve over $\mathbb{Q}(w)$ is given by

$$y^2 + xy + wy = x^3 - x^2 + (-554w + 1740)x + (-14641w - 9374). \tag{4.22}$$

This curve has conductor $\mathfrak{p}_2\mathfrak{p}_2'\mathfrak{p}_3^2$, but none of its twists over $\mathbb{Q}(w)$ have good reduction at $\mathfrak{p}_3$. It does have a twist of conductor $\mathfrak{p}_2\mathfrak{p}_2'\mathbb{Z}_{F_\infty}$ over $F_\infty$, which we take as a conjectural model.

Using isogenies of prime degree $\leq 50$, we encountered 8 elliptic curves in the $F_\infty$-isogeny class of (4.22). In the next section, we shall show that these curves indeed constitute the full $F_\infty$-isogeny class of $J_0^{\pm}(\mathcal{O}(1))$. The curves in this isogeny class whose $j$-invariant has valuation in $\{-15, -10\}$ are (4.22) and its $\mathrm{Gal}(F_\infty|F)$-conjugate. Therefore these conjugates give canonical models of $J^{\pm}(\Gamma)$.

*Remark* 4.5. The canonical $K'$-models $J_0(K')$ of $J(\mathcal{O}(1)^1)$ resulting from a choice of $K'$ as in (2.29) are all isomorphic. Indeed, we saw that there are 2-isogenies $J_0(K') \to J_0(\mathcal{O}(1))$ over $F_K$. But $J_0(\mathcal{O}(1))$ admits only one such isogeny.

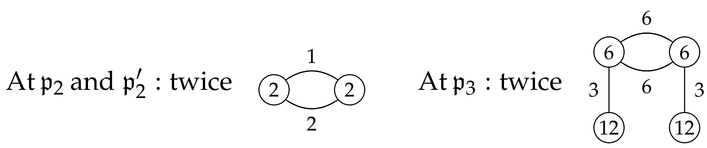
**e2d24D3:** Once more we have $|\mathrm{Cl}(\infty)| \neq 1$. $\Gamma^{(2)}$ generates a level $\mathfrak{p}_2^2$ order $\mathcal{O} = \mathbb{Z}_F[\Gamma^{(2)}]$ for which $\mathcal{O}^1 = \Gamma^{(2)}$. Let $\mathcal{O}(1)$ be an order containing $\mathcal{O}$. Table 3 shows that $[\mathcal{O}(1)^1 : \mathcal{O}^1] = 2$. By Corollary 1.3.5, this implies that we are in case (iii) of Proposition 1.1.2. Consequently $\mathrm{nrd}(\widehat{\mathcal{O}}^\times)$ equals $\mathbb{Z}_F^\times$, and Proposition 2.2.1 shows that $[\mathrm{P}\mathcal{O}^+ : \mathrm{P}\mathcal{O}^1] = 2$.

The signature $(1;2,2)$ of the group $\mathcal{O}^+$ was calculated by John Voight, using the methods from [Voi09b]. Combining the reasoning from the previous case with Lemma 2.3.5, we see that canonical models for $J^{\pm}(\Gamma)$ are given by Atkin–Lehner quotients of the Shimura curves $\mathrm{Sh}_0^{\pm}(\mathcal{O})$. These curves are defined over $F_\infty = F(w)$, where $w = w_{-2}$. As in the case e2d8D7, we settle for determining the isogeny class of $J_0(\mathcal{O})$. Once more we search $X_0(2)(F_\infty)$ using Algorithm 3.2.1. This yields a point in $X_0(2)(\mathbb{Q}(w))$ corresponding to the curve

$$y^2 + wxy + wy = x^3 + (w+1)x^2 + (3w+4)x + (2w+4). \tag{4.23}$$

*Remark* 4.6. Even though the level and discriminant are both Galois invariant in this case and the previous, we cannot use Theorem 2.1.7 to conclude that $j(J_0(\mathcal{O}))$ is rational since $|T(\mathcal{O}\infty)| \neq 1$.

**e2d33D12:** The group $\Gamma^{(2)}$ generates a maximal order $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma^{(2)}]$ whose norm 1 group is given by $\mathcal{O}(1)^1 = \Gamma^{(2)}$. Proposition 3.1.9 gives the following dual graphs for $\mathrm{Sh}(\mathcal{O}(1))$:




By Proposition 3.1.6(ii), the group $\mathcal{O}(1)^+$ has genus 1, hence signature $(1;2,2)$. As in the case e2d12D2, the connected components of $\mathrm{Sh}(\mathcal{O}(1))$ over $F_\infty = F(w_3)$ are given by $\mathrm{Sh}_0^{\pm}(\mathcal{O}(1))$. These components are isomorphic since $|T(\mathcal{O}\infty)| = 1$. For the same reason, Corollary 2.1.8 shows that $j(J_0(\mathcal{O}(1))) \in \mathbb{Q}$. Reasoning as in the case e2d13D4, we find the equation

$$y^2 + xy = x^3 + (\alpha+1)x^2 + (347\alpha - 1164)x + (-6063\alpha + 20448). \tag{4.24}$$

for $J_0(\mathcal{O}(1))$. As in the case e2d17D2, we see that the Atkin–Lehner quotient giving a canonical model of $J(\Gamma)$ over $F_\infty$ can be described by

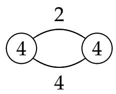
$$y^2 + xy = x^3 + (\alpha + 1)x^2 + (27\alpha - 84)x + (-63\alpha + 216). \tag{4.25}$$

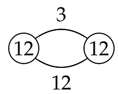
**e2d148D1i/ii/iii:** These three cases are conjugate (*cf.* Theorem 2.1.7); we calculate the first. As in the case e2d8D7, we only determine the isogeny class of the canonical model of $J(\Gamma)$.

Let $\mathcal{O} = \mathbb{Z}_F[\langle \Gamma^{(2)}, \alpha\beta \rangle]$. This is a level $\mathfrak{p}_2^4$ order satisfying $\mathcal{O}^1 = \langle \Gamma^{(2)}, \alpha\beta \rangle$. There exists a maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$ for which $\mathcal{O}(1)/\mathcal{O} \cong (\mathbb{Z}_F/\mathfrak{p}_2^2)^2$. Moreover, there exists a unique order $\mathcal{O}'$ inbetween $\mathcal{O}$ and $\mathcal{O}(1)$ for which $\mathcal{O}(1)/\mathcal{O}' \cong \mathbb{Z}_F/\mathfrak{p}_2 \times \mathbb{Z}_F/\mathfrak{p}_2^2$. As in Remark 1.2.2, one calculates $\mathrm{nrd}(\widehat{\mathcal{O}}'^\times) = \widehat{\mathbb{Z}}_F^\times$. An application of [Sij11, Algorithm 2.3] yields the equality $\mathcal{O}'^1 = \mathcal{O}^1$.

By Lemma 2.3.5, a canonical model of $X(\Gamma)$ can be obtained as an Atkin–Lehner quotient of the curve $\mathrm{Sh}_0(\mathcal{O}') = \mathrm{Sh}(\mathcal{O}')$. This curve is defined over $F_\infty = F$. As in the case e2d17D2, we see that $J_0(\mathcal{O})$ admits a 2-isogeny. Searching $X_0(2)$ using Algorithm 3.2.1, we end up with a conjectural model for $J_0(\mathcal{O})$ with conductor $\mathfrak{p}_2^3$ and $j$-invariant $2^6(-41\alpha^2 + 24\alpha + 141)$.

**e2d229D8:** These three cases are conjugate (*cf.* Theorem 2.1.7); we take the first. The order $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma^{(2)}]$ is maximal, and $\mathcal{O}(1)^1 = \langle \Gamma^{(2)}, \alpha\beta \rangle$. From Proposition 3.1.9, we get the following dual graphs for the Shimura curve $\mathrm{Sh}(\mathcal{O}(1))$:

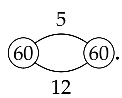
At $\mathfrak{p}_2$ : twice 

At $\mathfrak{p}_2'$ : twice 


Using Proposition 3.1.6, we conclude that the group $\mathcal{O}(1)^+$ has signature $(1;2)$. As in the case e2d12D2, the components $\mathrm{Sh}_0^\pm(\mathcal{O}(1))$ of $\mathrm{Sh}(\mathcal{O}(1))$ over the degree 2 extension $F_\infty$ of $F$ are isomorphic. We take $\mathrm{Sh}_0(\mathcal{O}(1))$ as a canonical model of $X(\Gamma)$.

As for the case e2d21D4, $J_0(\mathcal{O}(1))$ has a 2-isogeny over $F_\infty$, and as in the case e2d8D2, the isogeny factor $J_0(\mathcal{O}(1))$ of $J(\mathcal{O}(1))$ is defined over $F$ by [Hid81, Theorem 4.4]. Using Algorithm 3.2.1, we therefore searched $X_0(2)(F)$ instead of the larger set $X_0(2)(F_\infty)$. This gives rise to two conjectural models, one for each isogeny factor of $J(\mathcal{O}(1))$. Both have $j$-invariant $(n_2\alpha^2 + n_1\alpha + n_2)/2^{15}$, where $n_2 = 24628729701449988584212043$, $n_1 = 52087486182589166202672597$, and $n_0 = 11645298538324182916131980$. The conductors of these curves equal $\mathfrak{p}_2\mathfrak{p}_2'$, and they are isomorphic over $F_\infty$.

**e2d725D16i/ii:** These two cases are conjugate (*cf.* Theorem 2.1.7); we take the first. The group $\Gamma$ generates a maximal order $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma]$ for which $\Gamma = \mathcal{O}(1)^1$. We take $\mathrm{Sh}_0(\mathcal{O}(1)) = \mathrm{Sh}(\mathcal{O}(1))$ as a canonical model of $X(\Gamma)$; it is defined over $F_\infty = F$. The dual graph of $\mathrm{Sh}(\mathcal{O}(1))$ at $\mathfrak{p}_{17}$ obtained from Proposition 3.1.9 is given by

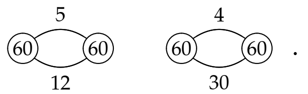



The traces of Frobenius of $J_0(\mathcal{O}(1))$ suggest that though $J_0(\mathcal{O}(1))[2](F)$ is empty, $J_0(\mathcal{O}(1))$ does admit a 17-isogeny. Browsing through the $F$-points of the curve $X_0(17)$, we end up with a candidate for $J_0(\mathcal{O}(1))$. It is an $F$-twist of the curve

$$y^2 + xy + w_5 y = x^3 + x^2 + (447w_5 - 4152)x + (-85116w_5 + 59004). \tag{4.26}$$

*Remark* 4.7. Note that the fact that $j(J_0(\mathcal{O}(1)))$ is in the fixed field $\mathbb{Q}(w_5)$ of $\mathrm{Aut}(F)$ also follows from Corollary 2.1.8 as $|\mathrm{Cl}(\infty)| = 1$.

**e2d1125D16:** We have $|\mathrm{Cl}(\infty)| \neq 1$. The group $\Gamma^{(2)}$ generates a maximal order $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma^{(2)}]$ for which $\mathcal{O}(1)^1 = \langle \Gamma^{(2)}, \beta \rangle$. Consider $\mathrm{Sh}(\mathcal{O}(1))$. At $\mathfrak{p}_2$, the Shimura curve $\mathrm{Sh}(\mathcal{O}(1))$ has the following dual graph by Proposition 3.1.9:




Hence $\mathcal{O}(1)^+$ has signature $(1;2)$. As in the case e2d21D4, the curves $\mathrm{Sh}_0^{\pm}(\mathcal{O}(1))$ give canonical models of $X^{\pm}(\Gamma)$. These Shimura curves are defined over $F_\infty = F(w)$, where $w = w_{-15}$. Poring over the traces of Frobenius of $J_0(\mathcal{O}(1))$, one suspects that this curve has a 17-isogeny. Searching through the subset $X_0(17)(\mathbb{Q}(w))$ of $X_0(17)(F_\infty)$, Algorithm 3.2.1 finds two $\mathrm{Gal}(F_\infty|F)$-conjugate conjectural models for $J^{\pm}(\Gamma)$ whose $j$-invariants equal $(53184785340479w \pm 30252086554835)/2^{34}$ and whose conductor equals $\mathfrak{p}_2\mathbb{Z}_{F_\infty}$.

*Remark* 4.8. As in the case e2d21D4, the canonical models $J_0^{\pm}(K')$ resulting from a choice of $K'$ in (2.29) for $K = \widehat{\mathcal{O}}(1)^\times$ are in fact independent of this choice. Note that there even exists an isomorphism $J(K') \cong J(\mathcal{O})$ over $F$. Indeed, the two models constructed above differ by a 2-isogeny over $F_\infty$.

**e3d1D6ii:** We know $j(J(\Gamma))$ from [Sij11]. The order $\mathcal{O} = \mathbb{Z}[\Gamma^{(2)}]$ is of level 4 and satisifies $\mathcal{O}^1 = \langle \Gamma^{(2)}, \alpha\beta \rangle$. Let $\mathcal{O}(1)$ be the maximal order containing $\mathcal{O}$. Then $\mathcal{O}(1)/\mathcal{O} \cong (\mathbb{Z}/2\mathbb{Z})^2$. We have $[\mathcal{O}(1)^1 : \mathcal{O}^1] = 4$ by [Sij11, Algorithm 2.3]. Hence we are in case (iii) of Proposition 1.1.4 by Corollary 1.3.5. Since $\mathrm{nrd}(\widehat{\mathcal{O}}^\times) = \widehat{\mathbb{Z}}^\times$, the canonical model $\mathrm{Sh}_0(\mathcal{O}) = \mathrm{Sh}(\mathcal{O})$ of $X(\mathcal{O}^1)$ is defined over $\mathbb{Q}_\infty = \mathbb{Q}$. By Lemma 2.3.5, there is an Atkin–Lehner quotient of $\mathrm{Sh}_0(\mathcal{O})$ giving a canonical model of $X(\Gamma)$. The remaining calculations are as in the case e2d1D6ii.

**e3d1D10:** The group $\Gamma^{(2)}$ generates a level $3^2$ order $\mathcal{O} = \mathbb{Z}[\Gamma^{(2)}]$ for which $\mathcal{O}^1 = \Gamma^{(2)}$. There exists a maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$ such that one has $\mathcal{O}(1)/\mathcal{O} \cong (\mathbb{Z}/3\mathbb{Z})^2$. Calculating the index $[O(1)^1 : \mathcal{O}^1]$ using [Sij11, Algorithm 2.3], we see that we are in case (iv) of Proposition 1.1.2. This implies that there is a unique level 3 Eichler order $\mathcal{O}(3)$ inbetween $\mathcal{O}$ and $\mathcal{O}(1)$ such that $\mathcal{O}^1 = \mathcal{O}(3)^1$. As in the case e2d1D6i, we see that we can take $J_0(\mathcal{O}(3)) = J(\mathcal{O}(3))$ as a canonical model of $J(\Gamma)$; it is defined over $\mathbb{Q}_\infty = \mathbb{Q}$. An equation for $J_0(\mathcal{O}(3))$ can be found in [GR06] and [Elk98].

**e3d12D3:** In this case $|\mathrm{Cl}(\infty)| \neq 1$. We know $j(J(\Gamma)) = 1728$ from [Sij11]. The group $\Gamma$ generates a level $\mathfrak{p}_3$ non-Eichler order $\mathcal{O}(\mathfrak{p}_3) = \mathbb{Z}_F[\Gamma]$ satisfying $\mathcal{O}(\mathfrak{p}_3)^1 = \Gamma$. Proposition 1.1.4 shows that

$$\mathrm{nrd}(\widehat{\mathcal{O}}(\mathfrak{p}_3)^\times) = U_{\mathfrak{p}_3}^2 \times \prod_{\mathfrak{p}\nmid 3} U_{\mathfrak{p}}^{(0)} = U_{\mathfrak{p}_3}^{(1)} \times \prod_{\mathfrak{p}\nmid 3} U_{\mathfrak{p}}^{(0)}. \tag{4.27}$$

Let $\kappa(\mathfrak{p}_3)$ be the residue field $\mathbb{Z}_F/\mathfrak{p}_3$. Since

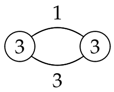
$$\mathrm{Ker}(\mathbb{Z}_F^+ \longrightarrow \kappa(\mathfrak{p}_3)^\times/\kappa(\mathfrak{p}_3)^{\times 2}) = \mathbb{Z}_F^{\times 2}, \tag{4.28}$$

we have $\mathrm{P}\mathcal{O}(\mathfrak{p}_3)^+ = \mathrm{P}\mathcal{O}(\mathfrak{p}_3)^1$ by Proposition 2.2.1. Therefore, as in the case e2d12D2, the two components $\mathrm{Sh}_0^{\pm}(\mathcal{O}(\mathfrak{p}_3))$ of $\mathrm{Sh}(\mathcal{O}(\mathfrak{p}_3))$ furnish canonical models of $X^{\pm}(\Gamma)$.

The curves $\mathrm{Sh}_0^{\pm}(\mathcal{O}(\mathfrak{p}_3))$ have canonical field of definition $F_{\mathfrak{p}_3\infty} = F_\infty = F(i)$. Indeed, since $\mathfrak{p}_3$ is non-trivial in $\mathrm{Cl}(\infty)$, the proof of Lemma 2.2.2 shows that the projection map $\mathrm{Cl}(\mathfrak{p}_3\infty) \to \mathrm{Cl}(\infty)$ is an isomorphism. As in the case e2d12D2, the curves $\mathrm{Sh}_0^{\pm}(\mathcal{O}(\mathfrak{p}_3))$ are isomorphic over $F_\infty$. The Jacobian $J_0(\mathcal{O}(\mathfrak{p}_3))$ is determined as in the case e2d1D6ii. It is given by

$$y^2 + (\alpha+1)xy + \alpha y = x^3 + (\alpha-1)x^2. \tag{4.29}$$

**e3d13D3i/ii:** We take the first of these two cases, which are conjugate by Theorem 2.1.7. $\Gamma^{(2)}$ generates a level $\mathfrak{p}_3'^2$ non-Eichler order $\mathcal{O} = \mathbb{Z}_F[\Gamma^{(2)}]$ such that $\mathcal{O}^1 = \langle \Gamma^{(2)}, \alpha\beta \rangle$. The index in Table 2, along with Corollary 1.3.5, shows that we are in case (iv) of Proposition 1.1.2. Hence there is a level $\mathfrak{p}_3'$ Eichler order $\mathcal{O}(\mathfrak{p}_3')$ inbetween $\mathcal{O}(1)$ and $\mathcal{O}$ for which $\mathcal{O}(\mathfrak{p}_3')^1 = \mathcal{O}^1$. We use $\mathrm{Sh}_0(\mathcal{O}(\mathfrak{p}_3')) = \mathrm{Sh}(\mathcal{O}(\mathfrak{p}_3'))$, defined over $F_\infty = F$, as a canonical model of $X(\mathcal{O}^1)$. A canonical model of $X(\Gamma)$ can be obtained by taking a suitable Atkin–Lehner quotient (*cf.* Lemma 2.3.5). Proposition 3.1.9 gives the following dual graph for $\mathrm{Sh}(\mathcal{O}(\mathfrak{p}))$ at $\mathfrak{p}_3$:

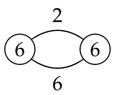



Searching $X_0(2)$ as in Algorithm 3.2.1 gives the following conjectural equation for $J_0(\mathcal{O}(\mathfrak{p}_3'))$:

$$y^2 + xy + y = x^3 + x^2 + (-190\alpha - 248)x + (1303\alpha + 1697). \tag{4.30}$$

Using prime isogenies of degree $\leq 50$, we found 12 elliptic curves in the isogeny class of (4.30). In the next section, we prove that these 12 curves constitute the full isogeny class of $J_0(\mathcal{O}(\mathfrak{p}_3'))$ over $F$.

It remains to recover the corresponding canonical model of $J(\Gamma)$. Although we could use Proposition 3.1.11 for this, we can also conclude by an ad hoc argument. Indeed, the non-trivial automorphism of the dual graph above gives rise to a genus 0 quotient graph. Therefore we conclude that the Atkin–Lehner automorphism giving rise to the canonical model of $X(\Gamma)$ acts trivially. Hence the dual graph of the corresponding quotient is given by




By Proposition 3.1.11, we see that in fact $X(\Gamma) \cong Y_0(\mathcal{O}(\mathfrak{p}_3'))/a(\mathfrak{p}_3)$.

The preceding shows that the canonical model of $J(\Gamma)$ is the unique elliptic curve that is 2-isogenous to $J_0(\mathcal{O}(\mathfrak{p}_3'))$ over $F$ and whose $j$-invariant has valuation $-8$ at $\mathfrak{p}_3$. An equation is given by

$$y^2 + xy + y = x^3 + x^2 + (495\alpha + 637)x + (9261\alpha + 12053). \tag{4.31}$$

**e3d21D3:** The group $\langle \Gamma^{(2)}, \alpha\beta \rangle$ is the norm 1 group of the level $\mathfrak{p}_3$ non-Eichler order $\mathcal{O}(\mathfrak{p}_3) = \mathbb{Z}_F[\langle \Gamma^{(2)}, \alpha\beta \rangle]$. Proposition 1.1.4 yields

$$\mathrm{nrd}(\widehat{\mathcal{O}}(\mathfrak{p}_3)^\times) = U_{\mathfrak{p}_3}^2 \times \prod_{\mathfrak{p} \nmid 3} U_{\mathfrak{p}}^{(0)} = U_{\mathfrak{p}_3}^{(1)} \times \prod_{\mathfrak{p} \nmid 3} U_{\mathfrak{p}}^{(0)}. \tag{4.32}$$

The curve $\mathrm{Sh}_0(\mathcal{O}(\mathfrak{p}_3))$ is defined over the degree 4 ray class extension

$$F \subsetneq F_\infty = F(\sqrt{-3}) \subsetneq F_{\mathfrak{p}_3\infty}. \tag{4.33}$$

As in the case e2d8D7, we settle for determining the isogeny class of a canonical model of $X(\Gamma)$. By Proposition 2.2.1, we have $[\mathrm{P}\mathcal{O}(\mathfrak{p}_3)^+ : \mathrm{P}\mathcal{O}(\mathfrak{p}_3)^1] = 2$. The group $\mathcal{O}(\mathfrak{p}_3)^+$ has signature $(0; 2, 2, 2, 2, 3)$: as in the case e2d24D3, this was calculated by John Voight. To construct the isogeny class of a canonical model of $X(\Gamma)$, we therefore have to choose groups $N'$ and $K'$ as in (2.29) for $K = \widehat{\mathcal{O}}(\mathfrak{p}_3)^\times$. As in the case e2d12D2, we have

$$N(K') \supset N(\widehat{\mathcal{O}}(\mathfrak{p}_3)^\times) = N(\widehat{\mathcal{O}}(1)^\times), \tag{4.34}$$

hence the $|T(K'\infty)| = 1$ by Proposition 1.4.2(iii). We conclude that the 4 components of $\mathrm{Sh}(K')$ over $F_{K'} = F_{\mathfrak{p}_3\infty}$ are isomorphic.

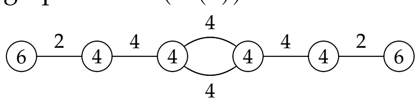
The discriminant of $B$ and the level $\mathfrak{p}_3$ of $\mathcal{O}(\mathfrak{p}_3)$ are Galois invariant. So if we choose the prime at which $N'$ is non-maximal to be Galois invariant as well (taking $N'$ to be non-maximal at the prime $\mathfrak{p}_2$ above 2, for example), then the proof of Corollary 2.1.8 shows that $j(J_0(K')) \in \mathbb{Q}$. Note that this conclusion holds regardless of the choice of $N'$, since the geometric components of $\mathrm{Sh}_0(K')$ are independent of the choice of $N'$ (*cf.* Proposition 2.2.3).

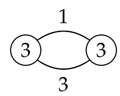
Explicitly, we now take $N'$ to be non-maximal at $\mathfrak{p}_2$. Proceeding as in the case e2d13D4, we get that the isogeny class of $J_0(K')$ is then given by the unique twist of conductor $\mathfrak{p}_3^2$ over $F_{\mathfrak{p}_3\infty}$ of the rational elliptic curve

$$y^2 + xy = x^3 - x^2 - 2x - 1. \tag{4.35}$$

*Remark* 4.9. A priori, the isogeny class of $J(\mathcal{O}(\mathfrak{p}_3)^1)$ obtained above depends on the choice of $N'$ in Lemma 2.2.2 and the resulting group $K'$ from (2.29). However, the fact that there is no factor $\mathfrak{p}_2$ in the conductor above leads one to suspect that the model is independent of the choice of $N'$. Though experimentally true, we have not been able to prove this fact.

**e3d28D18:** The order $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma^{(2)}]$ is maximal, and $\mathcal{O}(1)^1 = \Gamma^{(2)}$. Proposition 3.1.9 gives the following dual graphs for $\mathrm{Sh}(\mathcal{O}(1))$:

At $\mathfrak{p}_2$: twice 

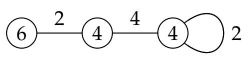

At $\mathfrak{p}_3$ and $\mathfrak{p}_3'$: twice 

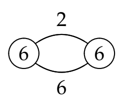

As in the case e2d21D4, we conclude that $\mathcal{O}(1)^+$ has signature $(1;3,3)$. The curve $\mathrm{Sh}_0(\mathcal{O}(1))$ is defined over $F_\infty = F(i)$. By Lemma 2.3.5, a canonical model for $X(\Gamma)$ is given by the unique $\mathbb{Z}/2\mathbb{Z}$ genus 1 Atkin–Lehner quotient of $\mathrm{Sh}_0(\mathcal{O}(1))$. Since we have $|T(\mathcal{O}(1)\infty)| = 1$ (*cf.* Proposition 1.4.2(iii)), we see that the components of $\mathrm{Sh}(\mathcal{O}(1))$ are isomorphic over $F_\infty$. Moreover, since the discriminant of $B$ is Galois invariant, Theorem 2.1.7 gives that $j(J_0(\mathcal{O}(1))) \in \mathbb{Q}$.

We can therefore proceed as in the case e2d13D4. Assisted by the dual graph above, we obtain that $J_0(\mathcal{O}(1))$ is the unique twist of conductor $\mathfrak{p}_2\mathfrak{p}_2'\mathfrak{p}_3\mathfrak{p}_3'$ over $F_\infty$ of the elliptic curve

$$y^2 + xy + y = x^3 + 2x + 32. \tag{4.36}$$

This curve has full 2-torsion over $F$. To determine the correct 2-isogeny, we calculate the dual graphs of the unique $\mathbb{Z}/2\mathbb{Z}$ genus 1 Atkin–Lehner quotient of $\mathrm{Sh}_0(\mathcal{O}(1))$:

Above 2: 


Above 3: 


This follows without further calculation from Proposition 3.1.11 considering that the corresponding Atkin–Lehner automorphism should act trivially on the dual graph at the primes above 3 (indeed, a non-trivial action leads to a genus 0 quotient). Note that in fact $X(\Gamma) \cong Y_0(\mathcal{O}(1))/a(\mathfrak{p}_2)$. We conclude that a canonical model for $J(\Gamma)$ is given by

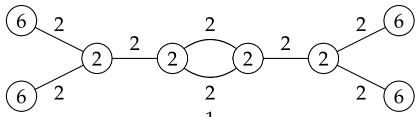
(4.37)
$$y^2 + \alpha xy + (\alpha+1)y = x^3 + (-\alpha-1)x^2 + (-944\alpha - 2496)x + (25532\alpha + 67552)$$

*Remark* 4.10. If we take

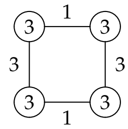
$$N' = U_{\mathfrak{p}_3}^{(1)} \times \prod_{\mathfrak{p} \nmid \mathfrak{p}_3} U_{\mathfrak{p}}^{(0)} \tag{4.38}$$

and $K = \widehat{\mathcal{O}}(1)^\times$ in (2.29), then another canonical model for $X(\Gamma)$ is given by $\mathrm{Sh}_0(K')$: indeed, we have $\mathrm{P}(K' \cap B^+) = \mathrm{P}\mathcal{O}(1)^1 = \Gamma^{(2)}$. Proposition 3.1.12 enables us to calculate the dual graphs of $\mathrm{Sh}(K')$:

At $\mathfrak{p}_2$: twice




At $\mathfrak{p}_3$ and $\mathfrak{p}_3'$: twice




This results in the same valuations, and hence the same canonical model, as above. Note that the graphs above do not depend on the choice of $K'$ in (2.29) by the final part of Proposition 3.1.12. Neither, as in the case e2d21D4, do the corresponding canonical models of $J_0(\mathcal{O}(1))$.

**e3d81D1:** We know $j(J(\Gamma)) = 0$ from [Sij11]. Let $\mathcal{O} = \mathbb{Z}_F[\Gamma]$. This is a level $\mathfrak{p}_3^3$ non-Eichler order for which $\mathcal{O}^1 = \Gamma$. There exists a maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$ such that $\mathcal{O}(1)/\mathcal{O} \cong \mathbb{Z}_F/\mathfrak{p}_3 \times \mathbb{Z}_F/\mathfrak{p}_3^2$. There is no order inbetween $\mathcal{O}$ and $\mathcal{O}(1)$ whose norm 1 group has genus 1 (this can also be deduced using Proposition 1.1.2). Therefore we take the curves $\mathrm{Sh}_0^\pm(\mathcal{O})$ as a canonical models of $X^\pm(\Gamma)$. As in Remark 1.2.2, one calculates

$$\mathrm{nrd}(\widehat{\mathcal{O}}^\times) = U_{\mathfrak{p}_3}^{(1)} \times \prod_{\mathfrak{p} \nmid 3} U_{\mathfrak{p}}^{(0)}. \tag{4.39}$$

The corresponding ray class field extension $F_{\mathfrak{p}_3\infty} = F(\sqrt{-3})$ is the canonical field of definition of $\mathrm{Sh}_0^\pm(\mathcal{O})$. Proceeding as in the case e2d1D6ii, we obtain the Weierstrass equation

$$y^2 + y = x^3 - 7 \tag{4.40}$$

for $J_0^+(\mathcal{O}) \cong J_0^-(\mathcal{O})$.

**e4d8D2i/iii:** We already encountered these two conjugate cases in [Sij11], where we failed to calculate them. Using the methods from this paper, the calculation of these curves is analogous to the case e2d17D2.

**e4d8D2ii:** In this case, the order $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma^{(2)}]$ is maximal and $\Gamma \subsetneq \mathcal{O}(1)^1$. We know $j(J^\pm(\Gamma))$ from [Sij11]. Moreover, we have that $F(j(J^\pm(\Gamma))) = F(i) = F_{\mathfrak{p}_2^2\infty}$. To obtain a model over a number field, we therefore pass to the non-trivial ray class field $F_{\mathfrak{p}_2^2\infty}$, over which $J^\pm(\Gamma)$ admits a model with conductor $\mathfrak{p}_2\mathfrak{p}_3^3$.

To find a canonical model for $X^\pm(\Gamma)$, we seek an isomorphism $X^\pm(\Gamma) \cong Y_0^\pm(K)$ for a suitable choice of $K$. By definition of the topology on $\widehat{B}^\times$, the cover $Y_0^\pm(K) \to$

$Y_0^{\pm}(\mathcal{O}(1))$ is the factorization of the Galois closure of some cover

$$Y_0^{\pm}(\mathcal{O}) \longrightarrow Y_0^{\pm}(\mathcal{O}(1)) \tag{4.41}$$

arising from a suborder $\mathcal{O}$ of $\mathcal{O}(1)$. Motivated by the conductor $\mathfrak{p}_2\mathfrak{p}_3^3$ above, we have tried orders $\mathcal{O}$ of level $\mathfrak{N} = \mathfrak{p}_2^i\mathfrak{p}_3^j$ with $i$ and $j$ small, proceeding as for the case e2d5D4ii. However, we did not obtain the cover $X^{\pm}(\Gamma) \to X^{\pm}(\mathcal{O}(1)^1) \cong Y_0^{\pm}(\mathcal{O}(1))$ as a factorization the Galois closure of (4.41). Consequently, we have not obtained a canonical model for $X^{\pm}(\Gamma)$. We formulate the following conjecture:

*Conjecture* 4.11. The group $\Gamma$ is not a congruence subgroup of $\mathcal{O}(1)^1$.

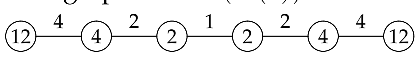
**e5d5D5i/ii:** These two cases are conjugate; we consider the first. Reasoning as in the case e2d1D6ii, one finds a canonical model of $J(\Gamma)$. As in the case e2d17D2, we can recover the curve $J_0(\mathcal{O}(\mathfrak{p}_{11}))$, which we did not calculate in [Sij11]. It is given by

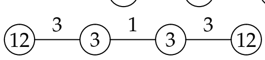
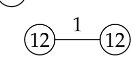
$$y^2 + (\alpha+1)xy + \alpha y = x^3 - \alpha x^2 + (-267\alpha - 166)x + (-2416\alpha - 1494). \tag{4.42}$$

**e5d5D5iii:** We determined $j(J(\Gamma))$ in [Sij11]. The order $\mathcal{O} = \mathbb{Z}_F[\Gamma^{(2)}]$ satisfies $\mathcal{O}^1 = \langle \Gamma^{(2)}, \alpha \rangle$. There exists a maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$ such that $\mathcal{O}(1)/\mathcal{O} \cong (\mathbb{Z}_F/\mathfrak{p}_2)^2$ as $\mathbb{Z}_F$-modules. The index from Table 2, along with Corollary 1.3.5, shows that we are in case (iii) of Proposition 1.1.2. By Lemma 2.3.5, there is an Atkin–Lehner quotient of $\mathrm{Sh}_0(\mathcal{O})$ giving a canonical model of $X(\Gamma)$. Since $\mathrm{nrd}(\widehat{\mathcal{O}}^{\times}) = \widehat{\mathbb{Z}}_F^{\times}$, we have that $\mathrm{Sh}_0(\mathcal{O}) = \mathrm{Sh}(\mathcal{O})$ is defined over $F_\infty = F$. As in the case e2d1D6ii, we obtain the following canonical model of $J(\Gamma)$:

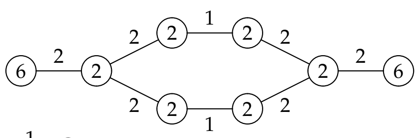
$$y^2 = x^3 + x^2 - 36x - 140. \tag{4.43}$$

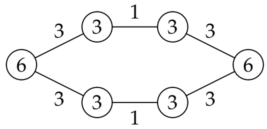
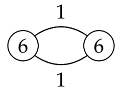
**e6d12D66i/ii:** We have $|\mathrm{Cl}(\infty)| \neq 1$. These two cases are conjugate (*cf.* Theorem 2.1.7); we consider the first. The order $\mathcal{O}(1) = \mathbb{Z}_F[\Gamma^{(2)}]$ is maximal and satisfies $\mathcal{O}(1)^1 = \Gamma^{(2)}$. The genus of the group $\mathcal{O}(1)^+$ equals 0. Indeed, Proposition 3.1.9 gives the following dual graphs for $\mathrm{Sh}(\mathcal{O}(1))$:

At $\mathfrak{p}_2$: twice 

At $\mathfrak{p}_3$: twice  At $\mathfrak{p}_{11}$ : twice 

Choose $K'$ as in (2.29) for $K = \widehat{\mathcal{O}}(1)^{\times}$. Then the canonical field of definition of $\mathrm{Sh}_0(K')$ is given by $F_{K'} = F_K = F_\infty = F(i)$. As in the case e2d12D2, we have $|T(K'\infty)| = 1$. By Lemmata 2.3.2 and 2.3.5, the isomorphic Jacobians $J_0^{\pm}(K')$ give canonical models for the curves $J^{\pm}(\Gamma)$. Proposition 3.1.12 gives the following dual graphs for $\mathrm{Sh}(K')$:

At $\mathfrak{p}_2$: twice 

At $\mathfrak{p}_3$: twice  At $\mathfrak{p}_{11}$ : twice 

As in the case e2d21D4, $J_0(K')$ admits a 2-isogeny for all $K'$ as in (2.29), and as in the case e2d8D2, $J_0(K')$ is defined over $F$ by [Hid81, Theorem 4.4]. Let us take

$$N' = U_{\mathfrak{p}_3}^{(1)} \times \prod_{\mathfrak{p} \nmid 3} U_{\mathfrak{p}}^{(0)} \tag{4.44}$$

in (2.29). Searching $X_0(2)(F)$ using Algorithm 3.2.1 then gives the conjectural model

$$y^2 + xy + (\alpha+1)y = x^3 + (\alpha-1)x^2 + (-405\alpha - 836)x + (4739\alpha + 7704). \quad (4.45)$$

It has conductor $\mathfrak{p}_2\mathfrak{p}_3^2\mathfrak{p}_{11}\mathbb{Z}_{F_\infty}$.

The choice of $N'$ affects the conductor of the resulting canonical $K'$-model of $J(\Gamma)$: for example, choosing $N'$ to be non-trivial at $\mathfrak{p}_2$, $\mathfrak{p}_{11}$ or $\mathfrak{p}'_{11}$, respectively, we get twists of (4.45) of conductor $\mathfrak{p}_2^4\mathfrak{p}_3\mathfrak{p}_{11}$, $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_{11}^2$, and $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_{11}\mathfrak{p}_{11}'^2$.

## 5. Proving correctness

This section considers the correctness of the candidate models of $J^{\pm}(\Gamma)$ obtained in the previous section. We combine the methods in [DD06], [DD08], and [SW05]. Throughout, we denote the absolute Galois group $\mathrm{Gal}(\overline{L}|L)$ of a number field $L$ by $G_L$.

**Correctness of the isogeny class of $E(\Gamma)$.** Let $B$ and $K$ be as in Section 0.3 and set $\mathfrak{D} = \mathfrak{D}(B)^f$. Choosing a simultaneous eigenvector of the Hecke operators $T_\mathfrak{p}$ from [Voi10], one obtains a system of eigenvalues

$$e = \{a(\mathfrak{p}) \mid \mathfrak{p} : B_\mathfrak{p} \text{ split and } K_\mathfrak{p} \text{ maximal}\}. \quad (5.1)$$

By [Hid81, Theorem 4.4], the systems of eigenvalues thus obtained are in bijection with the isogeny factors of $J(K)$ over $F$. Given a system $e$, the corresponding isogeny factor $A_e$ has real multiplication by the number field $M$ generated by the eigenvalues in $e$. Let $\lambda$ be a prime of $M$ over a rational prime $\ell$ and let

$$V_\lambda(A_e) = \varprojlim_n A_e[\lambda^n]. \quad (5.2)$$

be the $\lambda$-adic Tate module of $A_e$. Then we can construct a Galois representation

$$\rho(A_e)_\lambda : G_F \longrightarrow \mathrm{Aut}(V_\lambda(A_e)) \cong \mathrm{GL}_2(M_\lambda). \quad (5.3)$$

The representation $\rho(A_e)_\lambda$ is unramified outside the primes $\mathfrak{p}$ of $F$ that do not divide $\ell$ and at which $B_\mathfrak{p}$ is split and $K_\mathfrak{p}$ is maximal (*cf.* Theorem 2.1.6). Furthermore, if we let $\mathrm{Frob}(\mathfrak{p})$ be a Frobenius lift at such a $\mathfrak{p}$, then

$$\mathrm{tr}(\mathrm{Frob}(\mathfrak{p})) = a(\mathfrak{p}) \quad (5.4)$$

and

$$\det(\mathrm{Frob}(\mathfrak{p})) = \mathrm{nm}(\mathfrak{p}) = \epsilon_\ell(\mathrm{Frob}(\mathfrak{p})), \quad (5.5)$$

where $\epsilon_\ell : G_F \to \mathbb{Q}_\ell^\times$ is the $\ell$-adic cyclotomic character.

In the previous section, we have constructed conjectural models $E$ for $J_0(K)$ over $F_K$. These candidates $E$ are all isogenous to their $\mathrm{Gal}(F_K|F)$-conjugates, as required by Proposition 2.1.5. Consider the Weil restriction

$$A = \mathrm{Res}_{F_K|F}(E), \quad (5.6)$$

and let $A_0$ be an isogeny factor of $A$ over $F$. In all cases, the aforementioned isogenies can be used to show that $A_0$ has real multiplication by a number field $M$. For a prime $\lambda$ of $M$ over a rational prime $\ell$, we can consider the representation

$$\rho(A_0)_\lambda : G_F \longrightarrow \mathrm{Aut}(V_\lambda(A_0)) \cong \mathrm{GL}_2(M_\lambda). \quad (5.7)$$

This representation is unramified at the primes of $F$ that are coprime to $\ell$ and the conductor of $A_0$. The restriction of $\rho(A_0)_\lambda$ to the subgroup $G_{F_\infty}$ of $G_F$ is a direct sum of copies of the representation

$$\rho(E)_\ell : G_{F_\infty} \longrightarrow \mathrm{Aut}(V_\ell(E)) \cong \mathrm{GL}_2(\mathbb{Q}_\ell). \quad (5.8)$$

Conversely, one can recover $\rho(A_0)_\lambda$ from $\rho(E)_\ell$ as a factor of the induced representation

$$\mathrm{Ind}_{F_\infty|F}(\rho(E)_\ell) = \rho(A)_\ell : G_F \longrightarrow \mathrm{Aut}(V_\ell(A)). \tag{5.9}$$

By Faltings' isogeny theorem, proving that $E$ is in the isogeny class of $J_0(K)$ is therefore equivalent to proving that we have an isomorphism

$$\rho(A_0)_\lambda \cong \rho(A_e)_\lambda \tag{5.10}$$

for some isogeny factor $A_e$ of $J_0(K)$.

*Remark* 5.1. In all the cases in Section 4, we have $[F_K : F] \leq 2$ and therefore $\dim(A_0) \leq 2$ as well. Usually $F_K = F$, whence $A = A_0 = E$.

As in [SW05], we apply the following Theorem to show (5.10):

**Theorem 5.2** (Faltings-Serre, [Liv87])**.** *Let $M$ be a global field and let $S$ a finite set of primes of $M$. Let $M_S$ be the compositum of the quadratic extensions of $M$ unramified outside $S$. Suppose that $\rho_1, \rho_2 : G_M \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$ are continuous representations, unramified outside $S$, and furthermore satisfying*

(i) $\mathrm{tr}(\overline{\rho}_1) = 0 = \mathrm{tr}(\overline{\rho}_2)$ *and* $\det(\overline{\rho}_1) = \det(\overline{\rho}_2)$.

(ii) *There exist a set $P$ of primes of $M$, disjoint from $S$, for which*
   - *The image of* $\{\mathrm{Frob}(\mathfrak{p}) : \mathfrak{p} \in P\}$ *in the* $\mathbb{F}_2$*-vector space* $\mathrm{Gal}(M_S|M)$ *is non-cubic (that is, every cubic polynomial vanishing on $T$ vanishes on all of* $\mathrm{Gal}(M_S|M)$*); and*
   - *We have equalities*

$$\mathrm{tr}(\rho_1(\mathrm{Frob}(\mathfrak{p}))) = \mathrm{tr}(\rho_2(\mathrm{Frob}(\mathfrak{p}))) \tag{5.11}$$

*and*

$$\det(\rho_1(\mathrm{Frob}(\mathfrak{p}))) = \det(\rho_2(\mathrm{Frob}(\mathfrak{p}))) \tag{5.12}$$

*for all $\mathfrak{p}$ in $P$.*

*Then there exists an isomorphism of semi-simplified representations $\rho_1^{ss} \cong \rho_2^{ss}$.*

Using Theorem 5.2, it is often straightforward to show directly that

$$\rho(E)_2 \cong \rho(J_0(K))_2, \tag{5.13}$$

whence the correctness of the conjectural isogeny class given by $E$. Indeed, we take $S$ to equal the set of primes dividing the product of 2 and the conductor of $E$. After calculating the ray class group $R_S$ corresponding to $M_S$, one constructs a set $P$ of primes of $F_K$ mapping bijectively to the finite set $R_S$. For $\mathfrak{p}$ in $P$, one calculates $\mathrm{tr}(\rho(J_0(K))_2(\mathrm{Frob}(\mathfrak{p})))$ as in [Voi10], while $\mathrm{tr}(\rho(E)_2(\mathrm{Frob}(\mathfrak{p})))$ is easily calculated using the equation for $E$. Some elaborate calculations can be found at [Sij10b].

It remains to show the equalities

$$\mathrm{tr}(\overline{\rho}(J_0(K))_2(\mathrm{Frob}(\mathfrak{p}))) = 0 \tag{5.14}$$

and

$$\mathrm{tr}(\overline{\rho}(E)_2(\mathrm{Frob}(\mathfrak{p}))) = 0 \tag{5.15}$$

for all $\mathfrak{p}$. In all but four of the cases of the previous Section, follows from the fact that both $E$ and $J_0(K)$ admit a 2-isogeny over $F_K$, the latter of these coming from an Atkin–Lehner involution (*cf.* Lemma 2.3.5).

In the exceptional cases e2d725D16, e2d1125D16, e4d2624D4 and e5d725D25, the models $E$ and the curves $J_0(K)$ do admit a 2-isogeny. In principle, one could mimic the calculations in [SW05, Section 10.1] to prove correctness for these cases as well. Due to the extensive calculations involved, we have not looked further into this matter. However, in the case e2d1125D16, the existence of the isomorphism (5.10) follows as in [DD08] by noting that both $E$ en $J_0(K)$ admit a 17-isogeny over $F_K$ and applying the Jacquet-Langlands correspondence along with the results in [SW99].

**Correctness of the isomorphism class of $E(\Gamma)$.** In the Section 4, we have seen that Proposition 3.1.14 can often be used to determine $J_0(K)$ once the isogeny class of $J_0(K)$ over $F_K$ is known. Having found a curve $E$ isogenous to $J_0(K)$, we therefore proceed to calculate its full isogeny class.

Suppose we are not in one of the exceptional cases e2d725D16, e4d2624D4 and e5d725D25. Then if we let $\mathfrak{C}_o$ be the conductor of $E$, the same methods as above show that

$$\rho(A_0)_\lambda \cong \rho(f)_\lambda \tag{5.16}$$

for a Hilbert modular newform $f$ whose conductor $\mathfrak{C}$ satisfies $\mathfrak{C}_0 = \mathfrak{C}\mathbb{Z}_{F_K}$. Consider the reduction $\overline{\rho}(f)_\lambda : G_F \to \mathrm{GL}_2(\kappa_\lambda)$, where $\kappa_\lambda$ denotes the residue field of $M_\lambda$.

In the case where $A_0$ has complex multiplication, the isogeny class of $F_K$ is straightforward to calculate. In the non-CM cases, the proof of [DD06, Theorem 5.1] carries over word for word to all cases to show that the residual representations $\overline{\rho}(f)_\lambda$ are irreducible for all primes $\lambda$ of norm $> 50$. As a consequence, the elliptic curves $J_0(K)$ have no isogenies of prime degree $\ell > 50$. Indeed, a decomposition $\overline{\rho}(E)_\ell \cong \chi_1 \oplus \chi_2$ would give rise to a corresponding decomposition of the factors $\overline{\rho}(A_0)_\lambda = \overline{\rho}(f)_\lambda$ of (5.9).

In either case, we conclude that the conjectural isogeny classes of $J_0(K)$ that we constructed in the previous section are indeed complete.

## Appendix: Tables

This appendix consists of three tables detailing our results. As in [Sij11], we have assigned labels to the arithmetic $(1;e)$-groups $\Gamma$ in [Tak83, Theorem 4.1]. Such labels are of the form e$n_e$ d$n_d$ D$n_D$ r, where

- $n_e$ is the index of the unique elliptic point of $\Gamma$,
- $n_d$ is the discriminant of the center $F = \mathbb{Q}(\mathrm{tr}(\Gamma^{(2)}))$ of the quaternion algebra $B = F[\Gamma^{(2)}]$ associated to $\Gamma$,
- $n_D$ is the norm of the finite part of $\mathfrak{D}(B)^f$ of the discriminant of $B$ over $F$, and
- r is a roman numeral indicating the position at which $\Gamma$ occurs in [Tak83, Theorem 4.1] among the $\Gamma$ with the same $n_e$, $n_d$ and $n_D$.

Given an arithmetic $(1;e)$-group $\Gamma$, Table 1 describes

- The minimal polynomial $f^\alpha$ of a generator $\alpha$ of $F$,
- The Galois group $G$ of $F$ over $\mathbb{Q}$,
- The narrow class number $h^+$ of $F$ (the class number always equals 1),
- And the discriminant $\mathfrak{D}(B)$ of $B$.

In final column giving the discriminants, $\iota$ stands for an infinite place of $F$, and $\iota^c$ is the product of the other infinite places of $F$. For fixed $F$, the $\iota$ with a different number of primes $'$ are in different orbits under the action of $\mathrm{Aut}(F)$ on the set of infinite places of $F$.

Table 1: Fields and algebras associated to arithmetic $(1;e)$-groups

| Label | $f^\alpha$ | $G$ | $h^+$ | $\mathfrak{D}(B)$ |
|---|---|---|---|---|
| e2d1D6i | $t-1$ | $C_1$ | 1 | $\mathfrak{p}_2\mathfrak{p}_3$ |
| e2d1D6ii | $t-1$ | $C_1$ | 1 | $\mathfrak{p}_2\mathfrak{p}_3$ |
| e2d1D14 | $t-1$ | $C_1$ | 1 | $\mathfrak{p}_2\mathfrak{p}_7$ |
| e2d5D4i | $t^2-t-1$ | $C_2$ | 1 | $\mathfrak{p}_2\iota^c$ |
| e2d5D4ii | $t^2-t-1$ | $C_2$ | 1 | $\mathfrak{p}_2\iota^c$ |
| e2d5D4iii | $t^2-t-1$ | $C_2$ | 1 | $\mathfrak{p}_2\iota^c$ |
| e2d8D2 | $t^2-2$ | $C_2$ | 1 | $\mathfrak{p}_2\iota^c$ |
| e2d8D7i | $t^2-2$ | $C_2$ | 1 | $\mathfrak{p}_7\iota^c$ |
| e2d8D7ii | $t^2-2$ | $C_2$ | 1 | $\mathfrak{p}_7\iota'^c$ |
| e2d12D2 | $t^2-3$ | $C_2$ | 2 | $\mathfrak{p}_2\iota^c$ |
| e2d12D3 | $t^2-3$ | $C_2$ | 2 | $\mathfrak{p}_3\iota^c$ |
| e2d13D4 | $t^2-t-3$ | $C_2$ | 1 | $\mathfrak{p}_2\iota^c$ |
| e2d13D36 | $t^2-t-3$ | $C_2$ | 1 | $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_3'\iota^c$ |
| e2d17D2i | $t^2-t-4$ | $C_2$ | 1 | $\mathfrak{p}_2\iota^c$ |
| e2d17D2ii | $t^2-t-4$ | $C_2$ | 1 | $\mathfrak{p}_2\iota'^c$ |
| e2d21D4 | $t^2-t-5$ | $C_2$ | 2 | $\mathfrak{p}_2\iota^c$ |
| e2d24D3 | $t^2-6$ | $C_2$ | 2 | $\mathfrak{p}_3\iota^c$ |
| e2d33D12 | $t^2-t-8$ | $C_2$ | 2 | $\mathfrak{p}_2\mathfrak{p}_2'\mathfrak{p}_3\iota^c$ |
| e2d49D56 | $t^3-t^2-2t+1$ | $C_3$ | 1 | $\mathfrak{p}_2\mathfrak{p}_7\iota^c$ |
| e2d81D1 | $t^3-3t-1$ | $C_3$ | 1 | $\iota^c$ |
| e2d148D1i | $t^3-t^2-3t+1$ | $S_3$ | 1 | $\iota^c$ |
| e2d148D1ii | $t^3-t^2-3t+1$ | $S_3$ | 1 | $\iota'^c$ |
| e2d148D1iii | $t^3-t^2-3t+1$ | $S_3$ | 1 | $\iota''^c$ |
| e2d229D8i | $t^3-4t-1$ | $S_3$ | 2 | $\mathfrak{p}_2\mathfrak{p}_2'\iota^c$ |
| e2d229D8ii | $t^3-4t-1$ | $S_3$ | 2 | $\mathfrak{p}_2\mathfrak{p}_2'\iota'^c$ |
| e2d229D8iii | $t^3-4t-1$ | $S_3$ | 2 | $\mathfrak{p}_2\mathfrak{p}_2'\iota''^c$ |
| e2d725D16i | $t^4-t^3-3t^2+t+1$ | $D_4$ | 1 | $\mathfrak{p}_2\iota^c$ |
| e2d725D16ii | $t^4-t^3-3t^2+t+1$ | $D_4$ | 1 | $\mathfrak{p}_2\iota'^c$ |
| e2d1125D16 | $t^4-t^3-4t^2+4t+1$ | $C_4$ | 2 | $\mathfrak{p}_2\iota^c$ |
| e3d1D6i | $t-1$ | $C_1$ | 1 | $\mathfrak{p}_2\mathfrak{p}_3$ |
| e3d1D6ii | $t-1$ | $C_1$ | 1 | $\mathfrak{p}_2\mathfrak{p}_3$ |
| e3d1D10 | $t-1$ | $C_1$ | 1 | $\mathfrak{p}_2\mathfrak{p}_5$ |
| e3d1D15 | $t-1$ | $C_1$ | 1 | $\mathfrak{p}_3\mathfrak{p}_5$ |
| e3d5D5 | $t^2-t-1$ | $C_2$ | 1 | $\mathfrak{p}_5\iota^c$ |
| e3d5D9 | $t^2-t-1$ | $C_2$ | 1 | $\mathfrak{p}_3\iota^c$ |
| e3d8D9 | $t^2-2$ | $C_2$ | 1 | $\mathfrak{p}_3\iota^c$ |
| e3d12D3 | $t^2-3$ | $C_2$ | 2 | $\mathfrak{p}_3\iota^c$ |
| e3d13D3i | $t^2-t-3$ | $C_2$ | 1 | $\mathfrak{p}_3\iota^c$ |
| e3d13D3ii | $t^2-t-3$ | $C_2$ | 1 | $\mathfrak{p}_3\iota'^c$ |
| e3d17D36 | $t^2-t-4$ | $C_2$ | 1 | $\mathfrak{p}_2\mathfrak{p}_2'\mathfrak{p}_3\iota^c$ |
| e3d21D3 | $t^2-t-5$ | $C_2$ | 2 | $\mathfrak{p}_3\iota^c$ |
| e3d28D18 | $t^2-7$ | $C_2$ | 2 | $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_3'\iota^c$ |
| e3d49D1 | $t^3-t^2-2t+1$ | $C_3$ | 1 | $\iota^c$ |
| e3d81D1 | $t^3-3t-1$ | $C_3$ | 1 | $\iota^c$ |
| e4d8D2i | $t^2-2$ | $C_2$ | 1 | $\mathfrak{p}_2\iota^c$ |
| e4d8D2ii | $t^2-2$ | $C_2$ | 1 | $\mathfrak{p}_2\iota^c$ |
| e4d8D2iii | $t^2-2$ | $C_2$ | 1 | $\mathfrak{p}_2\iota'^c$ |
| e4d8D7i | $t^2-2$ | $C_2$ | 1 | $\mathfrak{p}_7\iota^c$ |
| e4d8D7ii | $t^2-2$ | $C_2$ | 1 | $\mathfrak{p}_7\iota'^c$ |
| e4d8D98 | $t^2-2$ | $C_2$ | 1 | $\mathfrak{p}_2\mathfrak{p}_7\mathfrak{p}_7'\iota^c$ |
| e4d2304D2 | $t^4-4t^2+1$ | $C_4$ | 2 | $\mathfrak{p}_2\iota^c$ |

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Table 1 – Continued from the previous page

| Label | $f^\alpha$ | $G$ | $h^+$ | $\mathfrak{D}(B)$ |
|---|---|---|---|---|
| e4d2624D4i | $t^4 - 2t^3 - 3t^2 + 2t + 1$ | $D_4$ | 1 | $\mathfrak{p}_2\iota^c$ |
| e4d2624D4ii | $t^4 - 2t^3 - 3t^2 + 2t + 1$ | $D_4$ | 1 | $\mathfrak{p}_2\iota'^c$ |
| e5d5D4 | $t^2 - t - 1$ | $C_2$ | 1 | $\mathfrak{p}_2\iota^c$ |
| e5d5D5i | $t^2 - t - 1$ | $C_2$ | 1 | $\mathfrak{p}_5\iota^c$ |
| e5d5D5ii | $t^2 - t - 1$ | $C_2$ | 1 | $\mathfrak{p}_5\iota^c$ |
| e5d5D5iii | $t^2 - t - 1$ | $C_2$ | 1 | $\mathfrak{p}_5\iota'^c$ |
| e5d5D180 | $t^2 - t - 1$ | $C_2$ | 1 | $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_5\iota^c$ |
| e5d725D25i | $t^4 - t^3 - 3t^2 + t + 1$ | $D_4$ | 1 | $\mathfrak{p}_5\iota^c$ |
| e5d725D25ii | $t^4 - t^3 - 3t^2 + t + 1$ | $D_4$ | 1 | $\mathfrak{p}_5\iota'^c$ |
| e5d1125D5 | $t^4 - t^3 - 4t^2 + 4t + 1$ | $C_4$ | 2 | $\mathfrak{p}_5\iota^c$ |
| e6d12D66i | $t^2 - 3$ | $C_2$ | 2 | $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_{11}\iota^c$ |
| e6d12D66ii | $t^2 - 3$ | $C_2$ | 2 | $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_{11}\iota'^c$ |
| e7d49D1 | $t^3 - t^2 - 2t + 1$ | $C_3$ | 1 | $\iota^c$ |
| e7d49D91i | $t^3 - t^2 - 2t + 1$ | $C_3$ | 1 | $\mathfrak{p}_7\mathfrak{p}_{13}\iota^c$ |
| e7d49D91ii | $t^3 - t^2 - 2t + 1$ | $C_3$ | 1 | $\mathfrak{p}_7\mathfrak{p}_{13}\iota'^c$ |
| e7d49D91iii | $t^3 - t^2 - 2t + 1$ | $C_3$ | 1 | $\mathfrak{p}_7\mathfrak{p}_{13}\iota''^c$ |
| e9d81D51i | $t^3 - 3t - 1$ | $C_3$ | 1 | $\mathfrak{p}_3\mathfrak{p}_{17}\iota^c$ |
| e9d81D51ii | $t^3 - 3t - 1$ | $C_3$ | 1 | $\mathfrak{p}_3\mathfrak{p}_{17}\iota'^c$ |
| e9d81D51iii | $t^3 - 3t - 1$ | $C_3$ | 1 | $\mathfrak{p}_3\mathfrak{p}_{17}\iota''^c$ |
| e11d14641D1 | $t^5 - t^4 - 4t^3 + 3t^2 + 3t - 1$ | $C_5$ | 1 | $\iota^c$ |

Let $\Gamma$ be a $(1;e)$-group and let $B$ be the corresponding quaternion algebra. Table 2 describes

- The orders $\mathcal{O}$ of $B$ generated by the groups $G$ inbetween $\Gamma^{(2)}$ and $\Gamma$,
- The norm 1 groups $\mathcal{O}^1$ associated to these orders; if this is not given by a $G$ as above, then the minimum of the indices $[\mathcal{O}^1 : G]$ is given,
- The level of $\mathcal{O}$,
- Whether or not $\mathcal{O}$ is Eichler,
- Whether or not $\Gamma$ is commensurable with a triangle group,
- And the degree of the map $X(\mathcal{O}^1) \to X(\mathcal{O}(1)^1)$ for a maximal order $\mathcal{O}(1)$ containing $\mathcal{O}$.

Table 2: The orders $\mathbb{Z}_F[G]$ for $\Gamma^{(2)} \subseteq G \subseteq \Gamma$

| Label | $\mathcal{O}$ | $\mathcal{O}^1$ | Level | Eichler? | $\Delta$ ? | Deg |
|---|---|---|---|---|---|---|
| e2d1D6i | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_5$ | Y | Y | 6 |
| e2d1D6ii | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^3$ | N | Y | 6 |
| e2d1D14 | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle \Gamma^{(2)}, AB \rangle]$ | $\langle \Gamma^{(2)}, AB \rangle$ | $(1)$ | Y | N | 1 |
| e2d5D4i | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^2$ | N | Y | 20 |
| | $\mathbb{Z}_F[\langle \Gamma^{(2)}, B \rangle]$ | 2 | $\mathfrak{p}_2$ | N | Y | 5 |
| e2d5D4ii | all $\mathbb{Z}_F[G]$ | 5 | $(1)$ | Y | Y | 1 |
| e2d5D4iii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle \Gamma^{(2)}, A \rangle]$ | $\langle \Gamma^{(2)}, A \rangle$ | $\mathfrak{p}_3$ | Y | Y | 10 |
| e2d8D2 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^5$ | N | Y | 24 |
| | $\mathbb{Z}_F[\langle \Gamma^{(2)}, AB \rangle]$ | $\langle \Gamma^{(2)}, AB \rangle$ | $\mathfrak{p}_2^4$ | N | Y | 12 |
| e2d8D7i | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle \Gamma^{(2)}, B \rangle]$ | $\langle \Gamma^{(2)}, B \rangle$ | $\mathfrak{p}_2^2$ | N | N | 2 |
| e2d8D7ii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle \Gamma^{(2)}, A \rangle]$ | $\langle \Gamma^{(2)}, A \rangle$ | $\mathfrak{p}_2^2$ | N | N | 2 |
| e2d12D2 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^4$ | N | Y | 12 |
| | $\mathbb{Z}_F[\langle \Gamma^{(2)}, A \rangle]$ | $\langle \Gamma^{(2)}, A \rangle$ | $\mathfrak{p}_2^3$ | N | Y | 6 |
| | $\mathbb{Z}_F[\langle \Gamma^{(2)}, B \rangle]$ | $\langle \Gamma^{(2)}, B \rangle$ | $\mathfrak{p}_2^3$ | N | Y | 6 |

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Table 2 – Continued from the previous page

| Label | $\mathcal{O}$ | $\mathcal{O}^1$ | Level | Eichler? | $\Delta$ ? | Deg |
|---|---|---|---|---|---|---|
| | $\mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $\mathfrak{p}_2^3$ | N | Y | 6 |
| | $\mathbb{Z}_F[\Gamma]$ | $\Gamma$ | $\mathfrak{p}_2$ | N | Y | 3 |
| e2d12D3 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^4$ | N | Y | 6 |
| e2d13D4 | all $\mathbb{Z}_F[G]$ | $\Gamma$ | (1) | Y | N | 1 |
| e2d13D36 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | (1) | Y | N | 1 |
| e2d17D2i | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2'^4$ | N | N | 6 |
| | $\mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $\mathfrak{p}_2'^2$ | N | N | 3 |
| e2d17D2ii | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2'^4$ | N | N | 6 |
| | $\mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $\mathfrak{p}_2'^2$ | N | N | 3 |
| e2d21D4 | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, B\rangle]$ | $\langle\Gamma^{(2)}, B\rangle$ | (1) | Y | N | 1 |
| e2d24D3 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^2$ | N | N | 2 |
| e2d33D12 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | (1) | Y | N | 1 |
| e2d49D56 | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, A\rangle]$ | $\langle\Gamma^{(2)}, A\rangle$ | (1) | Y | N | 1 |
| e2d81D1 | all $\mathbb{Z}_F[G]$ | $\Gamma$ | $\mathfrak{p}_2$ | Y | Y | 9 |
| e2d148D1i | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^5$ | N | N | 12 |
| | $\mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $\mathfrak{p}_2^4$ | N | N | 6 |
| e2d148D1ii | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^5$ | N | N | 12 |
| | $\mathbb{Z}_F[\langle\Gamma^{(2)}, B\rangle]$ | $\langle\Gamma^{(2)}, B\rangle$ | $\mathfrak{p}_2^4$ | N | N | 6 |
| e2d148D1iii | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^5$ | N | N | 12 |
| | $\mathbb{Z}_F[\langle\Gamma^{(2)}, A\rangle]$ | $\langle\Gamma^{(2)}, A\rangle$ | $\mathfrak{p}_2^4$ | N | N | 6 |
| e2d229D8i | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | (1) | Y | N | 1 |
| e2d229D8ii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | (1) | Y | N | 1 |
| e2d229D8iii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | (1) | Y | N | 1 |
| e2d725D16i | all $\mathbb{Z}_F[G]$ | $\Gamma$ | (1) | Y | N | 1 |
| e2d725D16ii | all $\mathbb{Z}_F[G]$ | $\Gamma$ | (1) | Y | N | 1 |
| e2d1125D16 | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, B\rangle]$ | $\langle\Gamma^{(2)}, B\rangle$ | (1) | Y | N | 1 |
| e3d1D6i | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_7$ | Y | Y | 8 |
| e3d1D6ii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $\mathfrak{p}_2^2$ | N | Y | 4 |
| e3d1D10 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_3^2$ | N | N | 4 |
| e3d1D15 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^2$ | N | N | 2 |
| | $\mathbb{Z}_F[\langle\Gamma^{(2)}, B\rangle]$ | $\langle\Gamma^{(2)}, B\rangle$ | (1) | Y | N | 1 |
| e3d5D5 | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, A\rangle]$ | $\langle\Gamma^{(2)}, A\rangle$ | $\mathfrak{p}_3$ | Y | Y | 10 |
| e3d5D9 | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $\mathfrak{p}_2$ | Y | Y | 5 |
| e3d8D9 | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, A\rangle]$ | $\langle\Gamma^{(2)}, A\rangle$ | $\mathfrak{p}_2^2$ | N | N | 2 |
| | other $G$ | $\Gamma$ | (1) | Y | N | 1 |
| e3d12D3 | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $\mathfrak{p}_2^2\mathfrak{p}_3$ | N | Y | 4 |
| | other $G$ | $\Gamma$ | $\mathfrak{p}_3$ | N | Y | 2 |
| e3d13D3i | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $\mathfrak{p}_3'^2$ | N | N | 4 |
| e3d13D3ii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, B\rangle]$ | $\langle\Gamma^{(2)}, B\rangle$ | $\mathfrak{p}_3'^2$ | N | N | 4 |
| e3d17D36 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | (1) | Y | N | 1 |
| e3d21D3 | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $\mathfrak{p}_3$ | N | N | 2 |
| e3d28D18 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | (1) | Y | N | 1 |
| e3d49D1 | all $\mathbb{Z}_F[G]$ | $\Gamma$ | $\mathfrak{p}_3$ | Y | Y | 28 |
| e3d81D1 | all $\mathbb{Z}_F[G]$ | $\Gamma$ | $\mathfrak{p}_3^3$ | N | Y | 12 |
| e4d8D2i | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, A\rangle]$ | $\langle\Gamma^{(2)}, A\rangle$ | $\mathfrak{p}_{17}$ | Y | Y | 18 |
| e4d8D2ii | all $\mathbb{Z}_F[G]$ | 9 | (1) | Y | Y | 1 |
| e4d8D2iii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $\mathfrak{p}_{17}'$ | Y | Y | 18 |
| e4d8D7i | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^4$ | N | N | 6 |
| e4d8D7ii | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^4$ | N | N | 6 |
| e4d8D98 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | (1) | Y | N | 1 |

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Table 2 – Continued from the previous page

| Label | $\mathcal{O}$ | $\mathcal{O}^1$ | Level | Eichler? | $\Delta$ ? | Deg |
|---|---|---|---|---|---|---|
| e4d2304D2 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $\mathfrak{p}_2^4$ | N | Y | 12 |
| | $\mathbb{Z}_F[\langle\Gamma^{(2)}, A\rangle]$ | $\langle\Gamma^{(2)}, A\rangle$ | $\mathfrak{p}_2^3$ | N | Y | 6 |
| | $\mathbb{Z}_F[\langle\Gamma^{(2)}, B\rangle]$ | $\langle\Gamma^{(2)}, B\rangle$ | $\mathfrak{p}_2^3$ | N | Y | 6 |
| | $\mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $\mathfrak{p}_2^3$ | N | Y | 6 |
| | $\mathbb{Z}_F[\Gamma]$ | $\Gamma$ | $\mathfrak{p}_2$ | N | Y | 3 |
| e4d2624D4i | all $\mathbb{Z}_F[G]$ | $\Gamma$ | $(1)$ | Y | N | 1 |
| e4d2624D4ii | all $\mathbb{Z}_F[G]$ | $\Gamma$ | $(1)$ | Y | N | 1 |
| e5d5D5i | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, B\rangle]$ | $\langle\Gamma^{(2)}, B\rangle$ | $\mathfrak{p}_{11}$ | Y | Y | 12 |
| e5d5D5ii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, B\rangle]$ | $\langle\Gamma^{(2)}, B\rangle$ | $\mathfrak{p}_{11}'$ | Y | Y | 12 |
| e5d5D5iii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, A\rangle]$ | $\langle\Gamma^{(2)}, A\rangle$ | $\mathfrak{p}_2^2$ | N | Y | 12 |
| | other $G$ | 6 | $(1)$ | Y | Y | 1 |
| e5d5D9 | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $\mathfrak{p}_5$ | Y | Y | 6 |
| e5d5D180 | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $(1)$ | Y | N | 1 |
| e5d725D25i | all $\mathbb{Z}_F[G]$ | $\Gamma$ | $(1)$ | Y | N | 1 |
| e5d725D25ii | all $\mathbb{Z}_F[G]$ | $\Gamma$ | $(1)$ | Y | N | 1 |
| e5d1125D5 | all $\mathbb{Z}_F[G]$ | $\Gamma$ | $\mathfrak{p}_5$ | N | Y | 3 |
| e6d12D66i | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $(1)$ | Y | N | 1 |
| e6d12D66ii | $\mathbb{Z}_F[\Gamma^{(2)}]$ | $\Gamma^{(2)}$ | $(1)$ | Y | N | 1 |
| e7d49D1 | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, B\rangle]$ | $\langle\Gamma^{(2)}, B\rangle$ | $\mathfrak{p}_2\mathfrak{p}_7$ | Y | Y | 72 |
| e7d49D91i | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, B\rangle]$ | $\langle\Gamma^{(2)}, B\rangle$ | $(1)$ | Y | N | 1 |
| e7d49D91ii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, A\rangle]$ | $\langle\Gamma^{(2)}, A\rangle$ | $(1)$ | Y | N | 1 |
| e7d49D91iiii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, AB\rangle]$ | $\langle\Gamma^{(2)}, AB\rangle$ | $(1)$ | Y | N | 1 |
| e9d81D51i | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, B\rangle]$ | $\langle\Gamma^{(2)}, B\rangle$ | $(1)$ | Y | Y | 1 |
| e9d81D51ii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, A\rangle]$ | $\langle\Gamma^{(2)}, A\rangle$ | $(1)$ | Y | Y | 1 |
| e9d81D51iii | $\mathbb{Z}_F[\Gamma^{(2)}] = \mathbb{Z}_F[\langle\Gamma^{(2)}, A\rangle]$ | $\langle\Gamma^{(2)}, A\rangle$ | $(1)$ | Y | Y | 1 |
| e11d14641D1 | all $\mathbb{Z}_F[G]$ | $\Gamma$ | $\mathfrak{p}_{11}$ | Y | Y | 12 |

The final Table 3 summarizes the canonical models for the curves $X^{\pm}(\Gamma)$ determined in Section 4. For these models $E^{\pm}(\Gamma)$, the table specifies

- The canonical field of definition $F_\Gamma$ of $E^{\pm}(\Gamma)$,
- A minimal field of definition $M_\Gamma$ of $E^{\pm}(\Gamma)$ (as an abstract field, $M_\Gamma$ is unique in all cases),
- The $j$-invariant $j(E^{\pm}(\Gamma))$ of $E^{\pm}(\Gamma)$,
- The conductor $\mathfrak{C}(E^{\pm}(\Gamma))$ of $E^{\pm}(\Gamma)$ over $F_\Gamma$, and
- Whether or not we succeeded in proving the correctness of $E^{\pm}(\Gamma)$ or its isogeny class.

These data determine $E^{\pm}(\Gamma)$ in all cases. In the cases where multiple conjugate values for $j$ were obtained, we give these $j$-invariants as distinct embeddings $\iota(j)$ into $\mathbb{C}$ for a fixed element $j$ of $F_\Gamma$ (*cf.* Theorem 2.1.7).

Frequently, it was only possible to determine an isogeny class of curves over $F$. In these cases, we have used the $j$-invariant of smallest height in this isogeny class, even though our experimental evidence indicates that Shimura curves tend to have a $j$-invariant of rather large height.

We point out a few anomalies in the list:

(i) At e3d21D3, $K_{-1321}$ denotes the subfield of $F_{\mathfrak{p}_3\infty}$ of discriminant $-1321$. This subfield is uniquely determined up to isomorphism (though not as a subfield of $F_{\mathfrak{p}_3\infty}$).

(ii) The cases e4d8D2ii lacks some entries: this is because we have been unable to prove that the associated group is congruence. See the corresponding paragraph in Section 4.

(iii) Finally, for the case e6d12D66, the canonical model depends on the choice of a certain compact open subgroup $K'$ as in Lemma 2.2.2. We again refer to Section 4 for examples.

Table 3: The canonical models for $J(\Gamma)$

| Label | $F_\Gamma$ | $M_\Gamma$ | $j(E^\pm(\Gamma))$ | $\mathfrak{C}(E^\pm(\Gamma))$ | Proved? |
|---|---|---|---|---|---|
| e2d1D6i | $F$ | $F$ | $7^3 2287^3/2^6 3^2 5^6$ | $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_5$ | Y |
| e2d1D6ii | $F$ | $F$ | $2^4 13^3/3^2$ | $\mathfrak{p}_2^3\mathfrak{p}_3$ | Y |
| e2d1D14 | $F$ | $F$ | $5^3 11^3 31^3/2^3 7^6$ | $\mathfrak{p}_2\mathfrak{p}_7$ | Y |
| e2d5D4i | $F$ | $F$ | $2^4 17^3$ | $\mathfrak{p}_2^3$ | Y |
| e2d5D4ii | $F$ | $\mathbb{Q}$ | $5^1 211^3/2^{15}$ | $\mathfrak{p}_2\mathfrak{p}_5^2$ | Y |
| e2d5D4iii | $F$ | $F$ | $-269^3/2^{10}3^5$ | $\mathfrak{p}_2\mathfrak{p}_3$ | Y |
| e2d8D2 | $F_{\mathfrak{p}_2^2\infty}$ | $\mathbb{Q}$ | $2^6 3^3$ | $\mathfrak{p}_2^6$ | Y |
| e2d8D7i/ii | $F$ | $F$ | $\iota(j), \iota'(j)$ | $\mathfrak{p}_2^2\mathfrak{p}_7$ | Y |
| (isogeny class) | where $j = \alpha 2^{12}(2\alpha+1)/7$ | | | | |
| e2d12D2 | $F_\infty$ | $F$ | $0$ | $\mathfrak{p}_2^4$ | Y |
| e2d12D3 | $F_\infty$ | $\mathbb{Q}$ | $2^2 193^3/3^1$ | $\mathfrak{p}_2^3\mathfrak{p}_3$ | Y |
| e2d13D4 | $F$ | $F$ | $-29^3 41^3/2^{15}$ | $\mathfrak{p}_2$ | Y |
| e2d13D36 | $F$ | $F$ | $11^3 23831^3/2^{10}3^2$ | $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_3'$ | Y |
| e2d17D2i/ii | $F$ | $F$ | $\iota(j), \iota'(j)$ | $\mathfrak{p}_2\mathfrak{p}_2'$ | Y |
| | where $j = -(2\alpha+3)(\alpha-2)^6(2\alpha-3)^3(4\alpha-17)^3(8\alpha+23)^3/2^{12}$ | | | | |
| e2d21D4 | $F_\infty$ | $F_\infty$ | $\iota(j), \iota'(j)$ | $\mathfrak{p}_2\mathfrak{p}_2'$ | Y |
| | where $j = -(w_{-7}-1)^5 3^3(2w_{-7}-3)^3(98w_{-7}+213)^3/2^{15}$ | | | | |
| e2d24D3 | $F_\infty$ | $\mathbb{Q}(w_{-2})$ | $\iota(j), \iota'(j)$ | $\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}_3'$ | Y |
| (isogeny class) | where $j = -w_{-2}2^{12}(w_{-2}+1)^2(w_{-2}-3)^3/3^3$. | | | | |
| e2d33D12 | $F_\infty$ | $F$ | $5^3 31^3/2^6 3^3$ | $\mathfrak{p}_2\mathfrak{p}_2'\mathfrak{p}_3\mathfrak{p}_3'$ | Y |
| e2d49D56 | $F$ | $\mathbb{Q}$ | $-5^3 1637^3/2^{18}7^1$ | $\mathfrak{p}_2\mathfrak{p}_7$ | Y |
| e2d81D1 | $F$ | $\mathbb{Q}$ | $-3^2 5^3 101^3/2^{21}$ | $\mathfrak{p}_2$ | Y |
| e2d148D1i/ii/iii | $F$ | $F$ | $\iota(j), \iota'(j), \iota''(j)$ | $\mathfrak{p}_2^3$ | Y |
| (isogeny class) | where $j = (3\alpha^2-\alpha-12)^2 2^6$ | | | | |
| e2d229D8i/ii/iii | $F_\infty$ | $F$ | $\iota(j), \iota'(j), \iota''(j)$ | $\mathfrak{p}_2\mathfrak{p}_2'$ | Y |
| | where $j = (4\alpha^2+8\alpha+1)^5(\alpha+1)^9$<br>$(8\alpha^2-9\alpha-6)^3(184\alpha^2-927\alpha-3724)^3/2^{15}$ | | | | |
| e2d725D16i/ii | $F$ | $F$ | $\iota(j), \iota'(j)$ | $\mathfrak{p}_2$ | N |
| | where $j = (w_5-2)^5(2429w_5+33625)^3/2^{17}$ | | | | |
| e2d1125D16 | $F_\infty$ | $F_\infty$ | $\iota(j), \iota'(j)$ | $\mathfrak{p}_2\mathfrak{p}_2'$ | Y |
| | where $j = -(17\alpha-3)(\alpha-1)^8(16\alpha+269)^3(240\alpha+187)/2^{34}$ | | | | |
| e3d1D6i | $F$ | $F$ | $4993^3/2^2 3^8 7^4$ | $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_7$ | Y |
| e3d1D6ii | $F$ | $F$ | $2^1 47^3/3^8$ | $\mathfrak{p}_2^3\mathfrak{p}_3$ | Y |
| e3d1D10 | $F$ | $F$ | $7^3 127^3/2^2 3^6 5^2$ | $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_5$ | Y |
| e3d1D15 | $F$ | $F$ | $6841^3/3^8 5^2$ | $\mathfrak{p}_3\mathfrak{p}_5$ | Y |
| e3d5D5 | $F$ | $\mathbb{Q}$ | $-5281^3/3^{16}5$ | $\mathfrak{p}_3\mathfrak{p}_5$ | Y |
| e3d5D9 | $F$ | $F$ | $7949^3/2^5 3^{10}$ | $\mathfrak{p}_2\mathfrak{p}_3$ | Y |
| e3d8D9 | $F$ | $F$ | $-2^6 239^3/3^{10}$ | $\mathfrak{p}_3$ | Y |
| e3d12D3 | $F_\infty$ | $F$ | $2^6 3^3$ | $\mathfrak{p}_3^2$ | Y |
| e3d13D3i/ii | $F$ | $F$ | $\iota(j), \iota'(j)$ | $\mathfrak{p}_3\mathfrak{p}_3'$ | Y |
| | where $j = (\alpha+2)^2 5^3(9\alpha+11)^3(277\alpha+301)^3/3^8$ | | | | |
| e3d17D36 | $F$ | $F$ | $11^3 41^3 131^3/2^2 3^{10}$ | $\mathfrak{p}_2\mathfrak{p}_2'\mathfrak{p}_3$ | Y |

Continued on the next page

Table 3 – Continued from the previous page

| Label | $F_\Gamma$ | $M_\Gamma$ | $j(E^\pm(\Gamma))$ | $\mathfrak{C}(E^\pm(\Gamma))$ | Proved? |
|---|---|---|---|---|---|
| e3d21D3 (isogeny class) | $F_{\mathfrak{p}_3\infty}$ | $K_{-1321}$ | $3^3 5^3 17^3$ | $\mathfrak{p}_3^2$ | Y |
| e3d28D18 | $F_\infty$ | $F$ | $23^3 41^3/2^2 3^8$ | $\mathfrak{p}_2'\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_3'$ | Y |
| e3d49D1 (isogeny class) | $F$ | $\mathbb{Q}$ | $-2^{12}7^1/3^1$ | $\mathfrak{p}_3$ | Y |
| e3d81D1 | $F_{\mathfrak{p}_3\infty}$ | $\mathbb{Q}$ | $0$ | $\mathfrak{p}_3^4$ | Y |
| e4d8D2i/iii | $F$ | $F$ | $\iota(j), \iota'(j)$ | $\mathfrak{p}_2\mathfrak{p}_{17}$ | Y |
| | where $j = -(\alpha+1)(2\alpha+1)^3(3\alpha-1)^3(16\alpha+203)^3/2^9 17^3$ | | | | |
| e4d8D2ii | ? | ? | $\iota(j), \iota'(j)$ | ? | Y |
| | where $j = w_{-2}(w_{-2}-1)(w_{-2}+3)^3(4w_{-2}-3)^3(15w_{-2}-13)^3/2^{14}$ | | | | |
| e4d8D7i/ii (isogeny class) | $F$ | $F$ | $\iota(j), \iota'(j)$ | $\mathfrak{p}_2^3\mathfrak{p}_7$ | Y |
| | where $j = -(4\alpha+5)^2(4\alpha-21)^3/7^2$ | | | | |
| e4d8D98 | $F$ | $\mathbb{Q}$ | $5^3 11^3 2383^3/2^9 7^2$ | $\mathfrak{p}_2\mathfrak{p}_7\mathfrak{p}_7'$ | Y |
| e4d2304D2 | $F_\infty$ | $F$ | $0$ | $\mathfrak{p}_2^4$ | Y |
| e4d2624D4i/ii | $F$ | $F$ | $\iota(j), \iota'(j)$ | $\mathfrak{p}_2$ | N |
| | where $j = -(3w_2-4)(1253w_2-2997)^3/2^{18}$ | | | | |
| e5d5D5i/ii | $F$ | $F$ | $\iota(j), \iota'(j)$ | $\mathfrak{p}_5\mathfrak{p}_{11}$ | Y |
| | where $j = \alpha^{20}(\alpha-4)^6(2\alpha-9)^3(160\alpha-527)^3/5^6 11^3$ | | | | |
| e5d5D5iii | $F$ | $\mathbb{Q}$ | $-2^4 109^3/5^6$ | $\mathfrak{p}_2^2\mathfrak{p}_5$ | Y |
| e5d5D9 | $F$ | $\mathbb{Q}$ | $23^3 73^3/3^2 5^8$ | $\mathfrak{p}_3\mathfrak{p}_5$ | Y |
| e5d5D180 | $F$ | $\mathbb{Q}$ | $7^3 2287^3/2^6 3^2 5^6$ | $\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_5$ | Y |
| e5d725D25i/ii | $F$ | $F$ | $\iota(j), \iota'(j)$ | $\mathfrak{p}_5$ | N |
| | where $j = -2^{12}(w_5-3)(4979w_5-8159)^3/5^7$ | | | | |
| e5d1125D5 | $F_\infty$ | $F$ | $0$ | $\mathfrak{p}_5^2$ | Y |
| e6d12D66i/ii | $F_\infty$ | $F$ | $\iota(j), \iota'(j)$ | depends | Y |
| | where $j = (10\alpha+17)^2(459\alpha+7382)^3/2^5 3^7 11^2$ | | | | |
| e7d49D1 | $F$ | $\mathbb{Q}$ | $5^3 11^3 31^3/2^3 7^6$ | $\mathfrak{p}_2\mathfrak{p}_7$ | Y |
| e7d49D91i/ii/iii | $F$ | $F$ | $\iota(j), \iota'(j), \iota''(j)$ | $\mathfrak{p}_7\mathfrak{p}_{13}$ | Y |
| | where $j = (\alpha^2-\alpha-2)^{18}(\alpha+2)^2(4\alpha-7)^2(\alpha-3)^2$ $(\alpha^2-3\alpha-11)^3(89\alpha^2-2111\alpha-737)^3/7^6 13^2$ | | | | |
| e9d81D51i/ii/iii | $F$ | $F$ | $\iota(j), \iota'(j), \iota''(j)$ | $\mathfrak{p}_3\mathfrak{p}_{17}$ | Y |
| | where $j = (\alpha^2-3)^{13}(\alpha+2)(4\alpha+1)^2(\alpha-3)^2$ $(65\alpha-4361\alpha-5810)^3/7^6 13^2$ | | | | |
| e11d14641D1 | $F$ | $\mathbb{Q}$ | $-2^{12}31^3/11^5$ | $\mathfrak{p}_{11}$ | Y |

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