Humbert Surfaces and Shimura Curves

Jeroen Sijsling

Universiteit Utrecht

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Universiteit Utrecht

Outline

Preamble

Definition

A *complex torus* is a quotient V/Λ, where

- V is a complex vector space, of dimension g say;
- Λ is a lattice in V of rank $2g$, so the canonical map $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \to V$ is an $\mathbb{R}$-isomorphism.

Any complex torus is a complex manifold in a canonical way, and will be viewed as such. Algebraizable tori are called *abelian varieties*.

Definition

Let $X = V/\Lambda$ be a complex torus. A *Riemann form* on X is a Hermitian form H on V with $\Im H(\Lambda \times \Lambda) \subseteq \mathbb{Z}$.

The additive group of all Riemann forms on X is called the (analytic) *Néron-Severi group* $\mathrm{NS}(X)$ of X. $\mathrm{NS}(X)$ classifies the line bundles on X up to analytic equivalence. It has a cohomological description as $H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$.

Definition

Let $X = V/\Lambda$ be a complex torus. The *dual complex torus* of X, denoted by $\hat{X}$ is defined as $\hat{X} = \hat{V}/\hat{\Lambda}$, where

- $\hat{V} := \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$;
- $\hat{\Lambda} := \{\varphi \in \hat{V} | \Im\varphi(\Lambda) \subseteq \mathbb{Z}\}$.

$\hat{X}$ parametrizes the line bundles on X analytically equivalent to $\mathcal{O}_X$.
Note that any Riemann form H yields a map

$$f_H : V/\Lambda \to \hat{V}/\hat{\Lambda}, v \mapsto H(\cdot, v).$$

A *positive definite* Riemann form for which this map is an isomorphism will be called a *principal polarization*. Tori admitting such a form are algebraizable: for these, one can replace "analytic" by "algebraic" in the preceding statements.

From now on we consider "PPAV"s (X, P), where X is a complex torus and P is a principal polarization of X. Recall that all endomorphisms of complex tori lift to complex linear maps.

Definition

Consider an endomorphism T of X. The linear lift of T to V induces a map $\hat{T}$ on $\hat{X}$. The *Rosati involution* of T, denoted by $T^\dagger$, is the unique endomorphism making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{T^\dagger} & X \\ {\scriptstyle f_P}\downarrow & & \downarrow{\scriptstyle f_P} \\ \hat{X} & \xrightarrow[\hat{T}]{} & \hat{X} \end{array}$$

Concretely, the Rosati involution is just taking the Hermitian adjoint w.r.t. P.

The Rosati-involution is a positive definite involution on $\mathrm{End}(X)$. It has a fixgroup $\mathrm{End}^{\dagger}(X)$, whose elements are called the *symmetric* endomorphisms. This subgroup is not a subring in general.

We have an

Important isomorphism

$$\mathrm{NS}(X) \xrightarrow{\sim} \mathrm{End}^{\dagger}(X)$$

defined by associating to H the endomorphism T such that for all $u, v \in V$

$$H(u, v) = H_P(u, Tv).$$

Real multiplication

Definition

A PPAV (X, P) is said to admit *real multiplication* if $\mathrm{End}(X) \otimes \mathbb{Q}$ contains a totally real field of degree g over $\mathbb{Q}$, pointwise fixed under the Rosati involution.

Some side remarks:

- If $g = 2$, any simple PPAV with endomorphism ring larger than $\mathbb{Z}$ admits RM. This is not true for larger g.
- One prove that any g-dimensional *complex torus* whose endomorphism algebra contains a totally real field of degree g is algebraic.

The case $g = 2$

By our Important isomorphism, one way to detect RM is to look at $\mathrm{NS}(X)$. Recall the general description

$$\begin{aligned} \mathrm{NS}(X) &= H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z}) \\ &= \mathrm{Hom}^{1,1}_{\mathrm{alt}}(V,\mathbb{R}) \cap \mathrm{Hom}^2_{\mathrm{alt}}(\Lambda,\mathbb{Z}) \\ &\subset \mathrm{Hom}^2_{\mathrm{alt}}(V,\mathbb{R}) \end{aligned}$$

Let $E \in \mathrm{Hom}^2_{\mathrm{alt}}(V,\mathbb{R})$. Then

- $E \in \mathrm{Hom}^{1,1}_{\mathrm{alt}}(V,\mathbb{R}) \Leftrightarrow E(i\cdot, i\cdot) = E(\cdot,\cdot)$.
- $E \in \mathrm{Hom}^2_{\mathrm{alt}}(\Lambda,\mathbb{Z}) \Leftrightarrow E(\Lambda,\Lambda) \subseteq \mathbb{Z}$.

These forms are related to the H considered previously by $E = \Im H$.

In our case $g = 2$, there is an isomorphism

$$\bigwedge_{\mathbb{Z}}^{2} \Lambda \xrightarrow{\sim} \mathrm{Hom}^2_{\mathrm{alt}}(\Lambda, \mathbb{Z})$$

by associating to $\lambda \wedge \mu$ the morphism

$$\lambda \wedge \mu \wedge . : \bigwedge_{\mathbb{Z}}^{2} \Lambda \to \bigwedge_{\mathbb{Z}}^{4} \Lambda$$

and identifying $\bigwedge_{\mathbb{Z}}^{4} \Lambda$ with $\mathbb{Z}$.
One can extend this $\mathbb{R}$-linearly to an isomorphism

$$\bigwedge_{\mathbb{R}}^{2} V \xrightarrow{\sim} \mathrm{Hom}^2_{\mathrm{alt}}(V, \mathbb{R}).$$

Let $w \in \bigwedge_{\mathbb{R}}^2 V$, and let E_w be the corresponding element of $\mathrm{Hom}_{\mathrm{alt}}^2(V, \mathbb{R})$. Then one can show that

- $E_w(i\cdot, i\cdot) = E_w(\cdot, \cdot) \Leftrightarrow w \in \mathrm{Ker}(\bigwedge_{\mathbb{R}}^2 V \to \bigwedge_{\mathbb{C}}^2 V)$;
- $E_w(\Lambda, \Lambda) \subseteq \mathbb{Z} \Leftrightarrow w \in \bigwedge_{\mathbb{Z}}^2 \Lambda$.

Proposition

$$\mathrm{Rank}\ \mathrm{NS}(X) = \mathrm{Rank}\ \mathrm{Ker}(\bigwedge_{\mathbb{Z}}^2 \Lambda \to \bigwedge_{\mathbb{C}}^2 V)$$

For $g = 2$, this gives us a very simple and explicit way of seeing how much RM our (X, P) has.

Humbert relations

Humbert applied this specifically to the *moduli* of algebraizable tori of dimension 2.
Consider the space

$$\mathfrak{H}_2 := \{\Pi = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in M(2, \mathbb{C}) | \Im(\Pi) \text{ positive definite}\}.$$

To such a matrix, we can associate the abelian surface $\mathbb{C}^2/\Lambda$, where Λ is generated by the vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}, \begin{pmatrix} \tau_2 \\ \tau_3 \end{pmatrix}.$$

Identifying $\bigwedge_{\mathbb{C}}^2 \mathbb{C}^2$ with $\mathbb{C}$, we see that the kernel of our map

$$\bigwedge_{\mathbb{Z}}^{2} \Lambda \longrightarrow \bigwedge_{\mathbb{C}}^{2} \mathbb{C}^2$$

has rank larger than 1 if and only if the τ_i satisfy a *Humbert relation*.

Definition

A Humbert relation is a relation of the form

$$A\tau_1 + B\tau_2 + C\tau_3 + D(\tau_2^2 - \tau_1\tau_3) + E = 0,$$

with A, B, C, D, E in $\mathbb{Z}$.

So these relations detect RM.
The *discriminant* Δ of a Humbert relation as above is given by

$$\Delta = B^2 - 4AC - 4DE.$$

Projecting to the Siegel modular threefold

$$\mathcal{A}_2^{\mathrm{an}} := \mathrm{Sp}(4,\mathbb{Z})\backslash\mathfrak{H}_2,$$

one obtains the so-called *Humbert surfaces* in the moduli space of abelian varieties of dimension 2.
Facts:

- To any discriminant Δ corresponds a unique irreducible Humbert surface H_Δ.
- For non-square Δ, the H_Δ parametrize abelian surfaces admitting an endomorphism of discriminant Δ.
- Humbert surfaces of discriminant n^2 parametrize abelian surfaces allowing an n-isogeny to a product of elliptic curves.

The algebraic theory

Main question

How does one obtain algebraic equations for $H_\Delta \subset \mathcal{A}_2$?

This question was answered by Humbert using theta functions, and recast in modern language by Lange and Wilhelm.
Consider an PPAS $(A, \mathcal{P})$. In terms of line bundles, our previous isomorphism

$$NS(A) \xrightarrow{\sim} \mathrm{End}(A)^\dagger.$$

is given by

$$[\mathcal{L}] \mapsto (A \xrightarrow{f_\mathcal{L}} \widehat{A} \xrightarrow{f_\mathcal{P}^{-1}} A).$$

Here $f_\mathcal{L}(a) = t_a^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$.

Let us denote by $\mathcal{P}_\alpha$ the line bundle thus corresponding to the endomorphism α, determined up to algebraic equivalence. Note that this has little to do with $\alpha^*\mathcal{P}$!
Using Riemann-Roch-type formulas for ASs, one can show that the trace and the norm of α can be recovered as follows:

$$\begin{aligned} \mathrm{Tr}(\alpha) &= (\mathcal{P}, \mathcal{P}_\alpha), \\ \mathrm{Nm}(\alpha) &= \frac{1}{2}(\mathcal{P}_\alpha, \mathcal{P}_\alpha). \end{aligned}$$

Hence the discriminant of α rewrites as

$$(\mathcal{P}, \mathcal{P}_\alpha)^2 - 2(\mathcal{P}_\alpha, \mathcal{P}_\alpha).$$

So

Proposition

Symmetric endomorphisms with discriminant Δ correspond to algebraic equivalence classes of line bundles $\mathcal{L}$ satisfying

$$(\mathcal{P}, \mathcal{L})^2 - 2(\mathcal{L}, \mathcal{L}) = \Delta.$$

Using Nakai-Moishezon for ASs, we see that

$$|\mathcal{P}_\alpha| \text{ is ample} \iff \mathrm{Tr}(\alpha), \mathrm{Nm}(\alpha) > 0,$$

and another Riemann-Roch formula tells us that in that case

$$h^0(\mathcal{P}_\alpha) = \frac{1}{2}(\mathcal{P}_\alpha, \mathcal{P}_\alpha) = \mathrm{Nm}(\alpha).$$

Genus 2 curves

are relevant because $\overline{\mathcal{M}_2} = \mathcal{A}_2$. They can be described by equations

$$y^2 = \prod_{i=1}^{6}(x - r_i).$$

Such a curve has six *Weierstrass points* $w_i = (r_i, 0)$.
On the Jacobian $J = \mathrm{Jac}(C) \cong S^2C$, these can be used to describe the two-torsion points

$$P_{ij} = w_i + w_j - 2w_1$$

and certain embeddings of C:

$$C_{ij} = \{c + w_i + w_j - 3w_1 | c \in C\}.$$

We principally polarize J by $C_{11} =: \vartheta(C)$.

(16, 6)-configuration and Kummer surfaces

One can easily read off that every P_{ij} is contained in exactly 6 C_{ij} and, conversely, that every C_{ij} contains exactly 6 P_{ij}. In jargon, the P_{ij} and C_{ij} form a (16, 6)-*configuration*.
We can make this into a (16, 6)-configuration of points and planes in $\mathbb{P}^3$ because of the following

Theorem (Gonzalez-Dorrego)

Let $(A, \mathcal{P})$ be a PPAS. Then the morphism

$$A \xrightarrow{|2\mathcal{P}|} \mathbb{P}^3$$

factors as

$$A \to A/ < -1 > \hookrightarrow \mathbb{P}^3$$

if and only if $(A, \mathcal{P}) \cong (J(C), \vartheta(C))$ for some C.

The quotient $A/<-1>$ appearing in the previous theorem is called the *Kummer surface* of A. So we know that, starting with a genus 2 curve C, one gets a Kummer surface K_C that can be embedded in $\mathbb{P}^3$. Note:

- Under this projection, the sixteen two-torsion points become nodes.
- The hyperplane sections for the linear system $|2\vartheta(C)|$ are exactly the divisors $t_x\vartheta(C) + t_{-x}\vartheta(C)$. One gets 16 doubly intersecting hyperplanes by choosing x to be 2-torsion, and these correspond exactly to our C_{ij}.

By our previous considerations, these points and planes form a $(16, 6)$-configuration.
Amusing side remark: one can do moduli of curves this way (see G-D).

The crux

Choose a node, say $\overline{0}$, of K_C. Take a plane P_C not containing this node, and project the configuration through the node to P_C. This gives a planar configuration of 6 lines intersecting in 15 points (the *vertices*), which I will call a *Kummer configuration*.

The configurations obtained by varying C are not arbitrary: there is a conic touching all six lines, namely the projection of the tangent cone at $\overline{0}$. Actually, the Kummer surface can be reconstructed from the configuration by taking a ramified cover and desingularizing.

Humbert curves

Restricted to curves D on K_C, this projection is birational (at least in the cases we consider).
For the degree, this implies

$$\deg(\pi(D)) = \deg(D) - \mathrm{mult}_0(D),$$

and the genus does not change.
By construction, the projection has the following remarkable property:

Key fact

$D' := \pi(D)$ can only be tangent with multiplicity 2 to the lines in the configuration, except possibly at the vertices.

Let us call such a curve a *Humbert curve* for the Kummer configuration.

In the other direction, there is the following result:

Proposition

Let D' be a degree δ Humbert curve for a Kummer configuration. Then there is a curve D on the corresponding Kummer surface projecting birationally to D' and having the property that $\delta \leq \deg(D) \leq 2\delta$.

This is shown by intersecting $\pi^{-1}(D')$ with a suitable hypersurface of degree $2\delta - 3$ containing K_C in $\mathbb{P}^3$. The proof uses explicit equations for Kummer surfaces.

Configurations induced by endomorphisms

Let us recapitulate:

- An endomorphism α gives an algebraic equivalence class of line bundles Θ_α.
- The symmetric sections in this bundle yield a family of divisors on the Kummer surface.
- Hence a family of Humbert curves in the Kummer plane with special properties of tangence.

And conversely: we have seen that we can lift a family of special divisors all the way back to obtain line bundles on $J(C)$.

Main Theorem (Humbert/Lange/Wilhelm)

More precisely:

Theorem

The following are necessary conditions for $J(C)$ to allow multiplication by an endomorphism with determinant Δ:

- *If $\Delta = 8d^2 + 9 - 2k$, $k \in \{4, 6, 8, 10, 12\}$: There exists a rational Humbert curve of degree $2d$, passing through $k - 1$ vertices;*
- *If $\Delta = 8d(d + 1) + 9 - 2k$, $k \in \{4, 6, 8, 10, 12\}$: There exists a rational Humbert curve of degree $2d + 1$, passing through $k - 1$ vertices;*
- *. . . (Two more cases.)*

Conversely, if such a curve in the Kummer configuration exists, then there is extra multiplication with determinant $\leq \Delta$.

Sketch of the proof for $\Delta = 8d(d+1) + 9 - 2k$:

Start with an endomorphism α of discriminant Δ, and $\mathrm{Tr}(\alpha) = 4d + 1$. Then

$$(\mathcal{P}, \mathcal{P}_\alpha) = \mathrm{Tr}(\alpha) = 4d,$$

$$(\mathcal{P}_\alpha, \mathcal{P}_\alpha) = \mathrm{Nm}(\alpha) = 4d^2 + k - 4.$$

One can calculate that $h^0(\Theta_\alpha)$ contains $d^2 + 1$ even functions. These give rise to $d^2 + 1$ symmetric divisors of degree $4d$ on K_J. Use Taylor expansions to show that there exists a symmetric divisor in Θ_α that has multiplicity $2d$ at 0, then project from $\overline{0}$ to get the requested curve.

The converse follows by similar techniques, by lifting the configuration to a line bundle.

Begin with a Humbert curve D'' in P_J of the special form above. Lift it to a curve D' on K_J of degree between $2d$ and $4d$. Pull this divisor back to a symmetric divisor D on J, and let $\mathcal{L} = \mathcal{O}_J(D)$. Then, Lange and Wilhelm show, using a formula relating $g(D)$ and $(\mathcal{L}, \mathcal{L})$ and further careful bookkeeping, that

$$(\mathcal{L}, \Theta)^2 - 2(\mathcal{L}, \mathcal{L}) \leq \Delta,$$

which, as we have seen, implies the existence of real multiplication fixed by Rosati of determinant $\leq \Delta$.

The link to algebraic equations

Theorem

There are coordinates x, y, z in the configuration P_C for which the lines in the configuration take the form

$$y + 2a_1x + a_1^2z = 0, \qquad y = 0,$$

$$y + 2a_2x + a_2^2z = 0, \quad y + 2x + z = 0,$$

$$y + 2a_3x + a_3^2z = 0, \qquad z = 0,$$

if and only if C is isomorphic to

$$Y^2 = X(X-1)(X-a_1)(X-a_2)(X-a_3).$$

To prove this, consider the ramification of a theta divisor.

Sample equation

The explicit equation for H_5 dates back to Humbert himself. In terms of the invariants a_i, H_5 is given by the equation

$$4(a_1^2a_3 - a_2^2 + a_3^2(1-a_1) + a_2 - a_3)(a_1^2a_2a_3 - a_1a_2^2a_3)$$
$$= (a_1^2a_3(a_2+1) - a_2^2(a_1+a_3) + a_2a_3^2(1-a_1) + a_1(a_2-a_3))^2.$$

Of course, the complexity of these cases increases with the discriminant.

Applications

- Allows the calculation of (Atkin-Lehner quotients of) Shimura curves associated to a quaternion algebra over $\mathbb{Q}$. This is achieved by intersecting Humbert surfaces.
- This immediately gives an integral family for the so-called Picard-Fuchs equation on such a Shimura curve.
- In particular, using correspondences of Fuchsian groups, one hopes to determine all Lamé equations with a single singularity in this way.
- If the Shimura curve is associated to a quaternion algebra over an *extension* of $\mathbb{Q}$, this approach can't be used.