EXAMPLES OF MODULI IN GENUS 2 AND 3

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Summary

These notes describe the classical approach to moduli for curves of genus 2 and 3, following the papers [Igu60] en [GK06]. In the final section, an interesting link to Del Pezzo surfaces is described. These notes are very sketchy and vague, probably even wrong in places, so read at your own risk.

1. Invariant theory

We wish to study the varieties representing isomorphism classes of curves over, say, an algebraically closed field $\overline{k}$ of characteristic 0. First hyperelliptic curves: it does not get any simpler than this. Indeed, considering that giving a hyperelliptic curve is the same as giving its branch locus in $\mathbf{P}^1$,

$$\mathcal{M}_g^{\mathrm{hyp}} \cong (S_{2(g+1)} \cdot \mathrm{Aut}(\mathbf{P}^1)) \backslash (\mathbf{P}^1)^{2(g+1)}_{\mathrm{gen}},$$

where the suffix gen means that no two points should coincide (we use the fact that we work in characteristic 0 here). Using our knowledge of $\mathrm{Aut}(\mathbf{P}^1)$, we see that if we denote $\mathbf{P}^1_* = \mathbf{P}^1 \backslash \{0, 1, \infty\}$, this simplifies to

$$\mathcal{M}_g^{\mathrm{hyp}} \cong S_{2g+2} \backslash (\mathbf{P}^1_*)^{2g-1}.$$

We need a description of the quotient as a variety. Observes that $\mathcal{M}_g^{\mathrm{hyp}}$ is unirational for all g. Phrasing this differently: any genus 2 hyperelliptic curve can be obtained by substituting suitable values for the s, t, u in the assignment

$$(s, t, u) \rightsquigarrow y^2 = x(x-1)(x-s)(x-t)(x-u).$$

It remains to be described what the S_6-action does to these parameters.

Non-hyperelliptic genus 3 curves can be identified with smooth quartics in $\mathbf{P}^2$. The moduli variety of quartics embedded in $\mathbf{P}^2$ is obviously given by $\mathbf{P}^{14}$ (consider the vector of coefficients). Let U be the non-singular part of this moduli space. Then $\mathrm{Aut}(\mathbf{P}^2) = \mathrm{PGL}(3, \overline{k})$ acts on $\mathbf{P}^{14}$ by substitution on the corresponding ternary forms. So, denoting by $\mathcal{M}_3^{\mathrm{nh}}$ the moduli space of non-hyperelliptic curves, we have

$$\mathcal{M}_3^{\mathrm{nh}} \cong \mathrm{PGL}(3, \overline{k}) \backslash U,$$

which is certainly not as explicit as one would like. Again, we do have unirationality.

The question is (1) what varieties represent these quotients, and (2) how to add points to $\mathcal{M}_3^{\mathrm{nh}}$ to obtain $\mathcal{M}_3$. On an abstract level, Mumford developed his *geometric invariant theory* to deal with this. The theory is very well adapted to solving existence problems and problems regarding compactification; however, this GIT is also much too hard for me.

On a more elementary level, one can work with a black box with more modest prerequisites, to wit *classical invariant theory*. This theory is a 19th century apparatus for churning out invariants of a geometric object. These invariants give vital information about the object: indeed, they characterize it up to isomorphism. The invariants form a ring over our algebraically closed ground field $\overline{k}$, and it can be proved that this ring is finitely generated. Incidentally, it was his proof of this theorem that got Hilbert branded as a theologian by Gordan. A nice introduction to this theory, with some applications, is given in [Mes91].

Although this theory works quite nicely, there is a catch; more than one, even. First of all, classical invariant theory typically works only on an open subset of the moduli space. The reason for this is that the theory works by considering certain canonical morphisms associated to the curves in question, which are embeddings only generically. Another problem we need to solve is to determine the structure of this open set of the moduli space: that is, to describe all relations between the invariants, to see what values they can take, and to check when tuples of invariants correspond to equivalent curves. In general, this is a rather daunting task; indeed, the intractability of these problems makes one appreciate the abstract approach to existence using representable functors or GIT.

Now to actually get started.

Definition 1.1. *Let V be a vector space over $\mathbf{C}$ of dimension n. Then $\mathrm{Sym}(V^*)$ is a graded ring with a right action of $\mathrm{GL}(V)$. A* covariant *of d-ary forms of order m is a linear map*

$$\varphi : \mathrm{Sym}(V^*)_d \longrightarrow \mathrm{Sym}(V^*)_m$$

such that for some k

$$\varphi(F^\gamma) = \det(\gamma)^k \varphi(F)^\gamma.$$

Intuitively, a covariant associates polynomials to polynomials in a GL-equivariant way.

Similarly, $\mathrm{Sym}(V)$ is a graded ring, on which $\mathrm{GL}(V)$ on the right by the inversion of the transposition of the action on $\mathrm{Sym}(V^*)$. Denoting this action by $G \mapsto G^{\gamma_*}$, a contravariant *of d-ary forms of order m is a linear map*

$$\varphi : \mathrm{Sym}(V^*)_d \longrightarrow \mathrm{Sym}(V)_m$$

such that for some k

$$\varphi(F^\gamma) = \det(\gamma)^k \varphi(F)^{\gamma_*}.$$

A variant *is a co- or contravariant, and an* invariant *is a variant of order 0. This will be an element of $\mathrm{Sym}(\mathrm{Sym}(V^*)_d)^{\mathrm{SL}(V)}$. Note that an invariant will in general not be invariant under $\mathrm{GL}(n, \mathbf{C})$!*

Note that a variant is determined by what it does to the generic d-ary form, so we can view variants as forms again (over some transcendental extension of $\mathbf{C}$).

There are ways to make new variants out of old ones. Some examples::

- The pairing $V \times V^* \to \mathbf{C}$ gives rise to pairings (or differential operations)

  $$\mathrm{Sym}(V^*) \times \mathrm{Sym}(V) \longrightarrow \mathrm{Sym}(V^*).$$

  and

  $$\mathrm{Sym}(V^*) \times \mathrm{Sym}(V) \longrightarrow \mathrm{Sym}(V).$$

The wonderful thing is that substituting variants in similar pairings gives rise to new variants, the order of which is the difference of the original orders.

- Another operation, specific to ternary forms, is as follows. For a form φ, define $D(\varphi)$ to be half its Hessian, with adjoint $D(\varphi)^*$. Choose a dot product $(\ ,\ )$ for matrices. Then, given a covariant φ and a contravariant ψ, one can construct four invariants

$$\begin{aligned} J_{30}(\varphi,\psi) &:= \det(D(\varphi)) \\ J_{11}(\varphi,\psi) &:= (D(\varphi), D(\psi)) \\ J_{22}(\varphi,\psi) &:= (D(\varphi)^*, D(\psi)^*) \\ J_{03}(\varphi,\psi) &:= \det(D(\psi)) \end{aligned}$$

- For binary forms F, G, the coefficients of which are homogeneous polynomials of degree r and s say, give rise to their i-th *transvectant*

$$(F,G)^i(x,y) := \frac{(r-i)!(s-i)!}{r!s!}\left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial y_1 \partial x_2}\right)^i F(x_1,y_1)G(x_2,y_2)|_{(x_j,y_j)=(x,y)}.$$

The idea is that by starting with a few variants (including, for example, the generic d-ary form itself) and repeating these operations a number of times, one will end up with the complete ring of invariants. This is difficult in practice, but in our examples below, the answers have been found.

Note that since our invariants can be chosen to be expressions with rational coefficients, they are Galois equivariant. This means that the field of moduli of a curve is exactly the field generated by its moduli point. There are examples where the field of moduli is not a field of definition. For example, in the hyperelliptic case, one will in general have to take a quadratic extension of the field of moduli because every hyperelliptic curve has an involution. For details, see [Mes91]. And even if the curve is definable over its field of moduli, there will not necessarily be a unique model over it, what with the moduli space being coarse and all. Problems like these will occur no matter what definition of moduli is used.

1.1. **Genus 2.** Quite a lot is known in genus 2, even over $\mathbb{Z}$. The classic account is the original paper [Igu60]. It uses the old language by Weil, but this does not make it less readable (amazing as it may seem, people could do quite decent algebraic geometry before EGA).

In characteristic $\neq 2$, every hyperelliptic curve can be written in the form $y^2 = f(x)$, where f is a sextic. As we saw earlier, f is only determined up to the action of $\mathrm{Aut}(\mathbf{P}^1)$. Classical invariant theory of sextics can be used to define the moduli of such a sextic (and hence of the hyperelliptic curve). The algebra of invariants turns out to be generated by five moduli, namely

$$\begin{aligned} A &= (ff')_6, \\ B &= ((ff')_4((ff')_4)')_4, \\ C &= ((ff')_4((ff')_4((ff')_4)')_2)_4 \end{aligned}$$

and two more invariants D and R which are even more hopeless to write down. These invariants were found by Clebsch, and have coefficients in $\mathbf{Q}$. The problem that now confronts us is threefold: what are the relations between these moduli, what moduli do actually arise, and what happens in characteristic 2, where hyperelliptic forms do not allow an expression as above?

Igusa solved all these problems simultaneously by constructing the moduli space of hyperelliptic curves over $\mathbf{Z}$. The idea is to define a normal form of a hyperelliptic curve C that works over any ring, as follows. Choose a Weierstrass point P of the curve, and a non-Weierstrass point Q. Let $Q' \neq Q$ be such that $Q + Q' = K_C$. If we define $D = 3P + Q$, then the complete linear system $|D|$ has dimension 3, so we get a morphism

$$X \xrightarrow{|D|} \mathbf{P}^2.$$

Riemann-Roch gives that this morphism is almost an immersion: only P and Q are mapped to the same double point. Now we normalize: we put the double point at $(0 : 1 : 0)$ with tangents $X = 0$ and $Z = 0$, and take $Y = 0$ to be tangent to the image of Q'. This yields the following affine equation for our curve:

$$XY^2 + (1 + aX + bX^2)Y + X^2(c + dX + X^2).$$

Conversely, any curve of this form (or rather, its normalization) is hyperelliptic.

The sextic associated to the Weierstrass points of our normal form is given by

$$(1 + aX + bX^2)^2 - 4X^3(c + dX + X^2).$$

A logical step now is to modify A, B, C, D and R such that the values they assume on this sextic are polynomials in $\mathbf{Z}$ with content 1. Igusa does this by a subtle transformation of this old basis, defining 5 invariants J_2, J_4, J_6, J_8 and J_{10} that are polynomials in a, b, c, d with coefficients in $\mathbf{Z}$ and content 1 that also generate the old rings of invariants over fields of characteristic $\neq 2$. Now the miracle is that *Igusa's forms reduce modulo 2 to the invariants in that characteristic.* Igusa gives no intrinsic reason why this should be so.

To give an example: if the characteristic is not equal to 2, then the value of J_4 at the sextic $c_0X^5 - c_1X^4 + \ldots - c_5$ is given by

$$-2^{-3}(5^2c_0^2c_3c_5 - 15c_0^2c_4^2 - 15c_0c_1c_2c_5 + 7c_0c_1c_3c_4 + 2^{-1}c_0c_2^2c_4$$
$$-c_0c_2c_3^2 + 2^2c_1^3c_5 - c_1^2c_2c_4 - c_1^2c_3^2 + c_1c_2^2c_3 - 2^{-4}3c_2^4).$$

So we have five polynomials over $\mathbf{Z}$. J_{10} corresponds to the discriminant of a sextic, so it will not be zero. Furthermore, one can check that the ideal of relations between the J_{2i} is generated by $J_2J_6 - J_4^2 - 4J_8$. Two questions remain:

(1) Given $(j_2, \ldots, j_{10})$, does there exist a curve with these invariants?
(2) When are the sextics corresponding to, say, $(j_2, \ldots, j_{10})$ and $(j_2', \ldots, j_{10}')$ projectively equivalent?

These are quite non-trivial questions. To my knowledge, Igusa was the first to prove that the answer to the first question is affirmative.

The answer to the second question was known already in the nineteenth century: $(j_2, \ldots, j_{10})$ and $(j_2', \ldots, j_{10}')$ are projectively equivalent if and only if $j_{2i}' = r^{2i}j_{2i}$ for some non-zero r.

Putting all of this together, one obtains

Theorem 1.1. *The coarse moduli scheme of genus 2 curves over* $\mathbf{Z}$ *is given by*

$$\mathrm{Spec}(\mathbf{Z}[X_1, X_2, X_3, X_4]/(X_1X_3 - X_2^2 - 4X_4))^{\mathbf{Z}/5\mathbf{Z}},$$

where $1 \in \mathbf{Z}/5\mathbf{Z}$ *acts by sending* X_i *to* $\zeta_5^i X_i$.

There is some fine print I won't get into. One should see a monomial $X_1^{e_1}X_2^{e_2}X_3^{e_3}X_4^{e_4}$ as corresponding to the invariant $J_1^{e_1}J_2^{e_2}J_3^{e_3}J_4^{e_4}J_5^{e_5}$ with $e_1 + 2e_2 + 3e_3 + 4e_4 + 5e_5 = 0$, which is truly invariant under $\mathrm{GL}(2, \mathbf{C})$.

By essentially the same argument as in the first paragraph, we can show that this moduli scheme is unirational ("the generic point has characteristic 0"). Indeed, it was known already before Igusa that it is rational. The more intricate properties of the moduli space were only vaguely grasped before Igusa's results. To name a few:

- The singular locus of $\mathcal{M}_2$ is given by the rational curves $J_2 = J_6 = J_8 = 0$ and $2 = J_2 = J_6 = 0$. The points on the former curve are represented by curves of the form

  $$Y^2 = X(X-1)(X-1-\zeta_5)(X-1-\zeta_5-\zeta_5^2)(X-1-\zeta_5-\zeta_5^2-\zeta_5^3),$$

  at least when the characteristic does not equal 2. The points on the latter correspond to curves $Y^2 - Y = X^5 + \alpha X^3$ over a field of characteristic 2. The intersection corresponds to $Y^2 - Y = X^5$ over a field of characteristic 2.
- Looking at the tangent space of the singularity, one sees that $\mathcal{M}_2$ is not embeddable in $\mathbf{A}^k_{\mathbf{Z}}$ for k smaller than 10. An embedding into $\mathbf{A}^{10}_{\mathbf{Z}}$, though, is easily given, since our ring is obviously generated by

  $$J_2^5 J_{10}^{-1}, J_2^3 J_4 J_{10}^{-1}, J_2^2 J_6 J_{10}^{-1}, J_2 J_8 J_{10}^{-1}, J_4 J_6 J_{10}^{-1},$$

  $$J_4 J_6^2 J_{10}^{-2}, J_6^2 J_8 J_{10}^{-2}, J_6^5 J_{10}^{-3}, J_6 J_8^3 J_{10}^{-3}, J_8^5 J_{10}^{-4}.$$

  When working over a field of characteristic $\neq 2$, an embedding in affine space of dimension 8 can be constructed, and this is again the minimum dimension.

1.2. **Genus 3.** As we have seen, this moduli space splits up into a part consisting of nonsingular quartics (open and of dimension 6) and a part consisting of hyperelliptic curves (closed and of dimension 5). For both parts, classical invariant theory gives a reasonably explicit description, but the problem of 'gluing' these descriptions has apparently not been solved explicitly yet. Only existence has been proved by the abstract machinery mentioned before.

The determination of the complete ring of invariants of smooth quartics was accomplished by Dixmier and Ohno. There are six independent invariants due to Dixmier, which give a subring of finite index of the complete ring of invariants; Ohno later found six additional invariants to generate the complete ring of invariants, and gave all relations between these twelve invariants over $\mathbb{C}$, together with the normalizations of these invariants over $\mathbb{Z}$. Of course, these invariants are quite horrible to write out (the largest one has degree 21).

A good reference for this is the paper [GK06]. In this paper, the invariants are used in combination with a stratification by A.M. Vermeulen to find conditions on the Dixmier-Ohno invariants that have to hold when the curve has, say, 7 hyperflexes (points with a tangent line of multiplicity 4).

Let us note that, like its lower genus predecessors, $\mathcal{M}_3$ is rational over $\overline{k}$ (as are $\mathcal{M}_4$, $\mathcal{M}_5$ and $\mathcal{M}_6$). This was proved by P. Katsylo in a horrendous flurry of manipulations of the classical invariants. As the genus grows, the $\mathcal{M}_g$ will stop being rational: they will be 'of general type', which is more or less the opposite of rationality (quoth Mumford).

2. Some fun: Del Pezzo surfaces

The amusing connections in this section were explained to me by E. Looijenga. Because I am not very fluent in the theory of geometric surfaces, it is necessarily of a very sketchy nature. [Dem80] seems a reasonable reference. We again work over an algebraically closed field of characteristic 0.

Let us start with a general construction. Starting with $\mathbf{P}^2$, which has canonical sheaf $K_{\mathbf{P}^2}$ isomorphic to $-3H$, one can choose r points and blow them up. This gives a surface

$$X \xrightarrow{\pi} \mathbf{P}^2$$

with exceptional divisors $E_1, \dots, E_r$. It is natural to choose $\{\pi^* H, E_1, \dots, E_r\}$ as basis for $\mathrm{Pic}(X)$. With respect to this basis, the intersection product has matrix

$$\begin{pmatrix} 9-r & -1 & -1 & \dots & -1 & -1 \\ -1 & -1 & 0 & \dots & 0 & 0 \\ -1 & 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \dots & -1 & 0 \\ -1 & 0 & 0 & \dots & 0 & -1 \end{pmatrix}.$$

This form is negative definite on $K_X^{\perp}$. A natural question is to ask how to obtain all the other 'exceptional bases', *i.e.* how to describe the other sets

$$\{E_1', \dots, E_7'\}$$

that, together with K_X, form a basis of $\mathrm{Pic}(X)$ with intersection matrix as above. Let us call the set of these bases Exc_X. The following proposition answers our question:

Proposition 2.1. *Consider the group*

$$W_X := \mathrm{Aut}_{(\,,\,)}(\mathrm{Pic}(X), K_X).$$

of automorphisms of $\mathrm{Pic}(X)$ *fixing* K_X *and leaving the intersection pairing invariant. The set* Exc_X *is a torsor under this group.* W_X *is isomorphic to the Weyl group* $W(E_7)$.

In the case $r = 7$, there is a theorem which makes all this exceedingly relevant to us:

Theorem 2.2. *Let X be as above be obtained from seven points in 'general position'. The linear system K_X^{-1} gives a morphism $X \longrightarrow \mathbf{P}^2$ of degree 2 with a smooth quartic as its ramification locus.*

Conversely, given a smooth quartic C, construct the double covering X of $\mathbf{P}^2$ branched exactly along this quartic (an affine equation for this double covering is $Z^2 = f(X,Y)$, where f is an affine equation for C). Then X has 56 exceptional divisors, two for every bitangent of C. One can choose seven of these and blow X down to $\mathbf{P}^2$.

This gives a correspondence between smooth quartics and Del Pezzo surfaces.

In fact, this extends to a correspondence between smooth quartics with a level 2 structure and Del Pezzo surfaces X with a given element of Exc_X. Under this construction, such a surface corresponds to an ordered choice of seven points in general position in $\mathbf{P}^2$ up to automorphisms of $\mathbf{P}^2$. And in fact all these correspondences are algebraic.

So let $(\mathbf{P}^2)^7_{\mathrm{gen}}$ be the seven-tuples of points in $(\mathbf{P}^2)^7$ in general position. Then

$$\mathcal{M}_3^{\mathrm{nh}}(2) \cong \mathrm{Aut}(\mathbf{P}^2)\backslash(\mathbf{P}^2)^7_{\mathrm{gen}} \cong (\mathbf{P}^2_*)^3_{\mathrm{gen}},$$

and, using proposition 2.1,

$$\mathcal{M}_3^{\mathrm{nh}} \cong W_X\backslash\mathcal{M}_3^{\mathrm{nh}}(2) \cong (W(E_7)\cdot \mathrm{Aut}(\mathbf{P}^2))\backslash(\mathbf{P}^2)^7_{\mathrm{gen}} \cong W(E_7)\backslash(\mathbf{P}^2_*)^3_{\mathrm{gen}}.$$

In fact, one can show, using some geometric invariant theory, that a similar isomorphism holds on the entire $\mathcal{M}_3$ if we relax the generality conditions on our seven points. The resemblance with the situation in genus 2 is quite striking: it can be shown that there, too,

$$\mathcal{M}_2(2) \cong \mathrm{Aut}(\mathbf{P}^1)\backslash(\mathbf{P}^1)^6_{\mathrm{gen}} \cong (\mathbf{P}^1_*)^3_{\mathrm{gen}}.$$

This is no coincidence. Such 'point spaces' occur more frequently when considering moduli: *viz.* [DO88].

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